Orbital Free Entropy & Its Legendre Transform Approach

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(M, τ) = W^*-prob. sp.

M ← W^*-alg., τ ← f.n. tracial state

- $X = X^* \in M$ random variable
- $\mathbb{X} = (X_1, \dots, X_\ell)$ (ℓ-)tuple of random variables $X_j = X_j^* \in M$.

  $W^*(\mathbb{X})$:= the vN subalg. gen. by the X_j's in M.

M_N^{sa}, Leb on $M_N^{sa} \cong \mathbb{R}^{N^2}$.

$U(N)$, γ_N = the Haar prob. meas.

$$\Gamma_R(\mathbb{X}; N, m, \delta) = \Gamma_R(X_1, \ldots, X_\ell; N, m, \delta)$$

$$= \text{all } \mathbb{A} = (A_1, \ldots, A_\ell) \in (M_N^{sa})^\ell \ ;$$

$$|\mathrm{tr}_N(A_{j_1} \cdots A_{j_r}) - \tau(X_{j_1} \cdots X_{j_r})| < \delta$$

whenever $1 \leq j_1, \ldots, j_r \leq \ell, \ 1 \leq r \leq m.$

$\mathbb{X}_i = (X_{i1}, \ldots, X_{i\ell_i})$ given. $\quad \ell := \sum_{i=1}^n \ell_i .$

$$\Phi_N : U(N)^n \times (M_N^{sa})^\ell \longrightarrow (M_N^{sa})^\ell$$

$$((U_i)_{i=1}^n, (\mathbb{A}_i)_{i=1}^n) \longmapsto (U_i \mathbb{A}_i U_i^*)_{i=1}^n$$

$$U_i \mathbb{A}_i U_i^* = (U_i A_{ij} U_i^*)_{j=1}^{\ell_i}$$

<u>Orbital free entropy</u> :

$$\chi_{\mathrm{orb}}(\mathbb{X}_1, \dots, \mathbb{X}_n) := \sup_{R>0} \lim_{\substack{m\to\infty \\ \delta\to 0}} \overline{\lim_{N\to\infty}} \text{ of :}$$

$$\sup_{\mu} \frac{1}{N^2} \log \gamma_N^{\otimes n} \otimes \mu \left(\Phi_N^{-1}\left(\Gamma_R(\mathbb{X}; N, m, \delta)\right)\right).$$

μ ↑ prob. meas. on $(M_N^{sa})^{\ell}$

$$\mu_N = \frac{1}{\mathrm{Leb}\left(\prod_{i=1}^{n} \Gamma_R(X_i; N, m, \delta)\right)} \mathrm{Leb}\Big|_{\prod_{i=1}^{n} \Gamma_R(X_i; N, m, \delta)}$$

$$\longrightarrow \mathrm{Log}\, \gamma_N^{\otimes n} \otimes \mu_N \left(\Phi_N^{-1}\left(\Gamma_R(X; N, m, \delta)\right)\right)$$

$$= \mathrm{Log}\, \mathrm{Leb}\left(\Gamma_R(X; N, m, \delta)\right) - \sum_{i=1}^{n} \log \mathrm{Leb}\left(\Gamma_R(X_i; N, m, \delta)\right)$$

$$\text{"}\longleftrightarrow\text{"} \quad \chi_{orb}(X_1, \ldots, X_n) = \chi(X) - \sum_{i=1}^{n} \chi(X_i).$$

$\Gamma_{orb}(\mathbb{X}_1,\dots,\mathbb{X}_n : \mathbb{A}_1,\dots,\mathbb{A}_n ; N, m, \delta)$

$\mathbb{X} = \mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n$

$:=$ all $(U_i)_{i=1}^n \in \mathrm{U}(N)^n$;

$$(U_i \mathbb{A}_i U_i^*)_{i=1}^n = \left(\left(U_i A_{ij} U_i^*\right)_{j=1}^{\ell_i}\right)_{i=1}^n \in \Gamma(\mathbb{X}; N, m, \delta).$$

$$\gamma_N^{\otimes n} \otimes \mu\left(\Phi_N^{-1}\left(\Gamma_R(\mathbb{X}; N, m, \delta)\right)\right)$$

$$= \int_{\prod_{i=1}^n \Gamma_R(\mathbb{X}_i; N, m, \delta)} \gamma_N^{\otimes n}\left(\Gamma_{orb}\left((\mathbb{X}_i)_{i=1}^n : (\mathbb{A}_i)_{i=1}^n ; N, m, \delta\right)\right) d\mu$$

$d\mu$ w.r.t. $(\mathbb{A}_i)_{i=1}^n$

disintegration formula

Reformulation :

$$\chi_{orb}(\mathbb{X}_1,\dots,\mathbb{X}_n) = \sup_{R>0} \lim_{\substack{m\to\infty \\ \delta\downarrow 0}} \overline{\lim_{N\to\infty}} \text{ of :}$$

$$\sup_{\mathbb{A}_i \in \Gamma_R(\mathbb{X}_i; N, m, \delta)} \frac{1}{N^2} \log \gamma_N^{\otimes n}\left(\Gamma_{orb}(\mathbb{X}_1,\dots,\mathbb{X}_n : \mathbb{A}_1,\dots,\mathbb{A}_n ; N, m, \delta)\right)$$

General results / Major questions :

- $\chi_{orb} = 0 \longleftrightarrow$ free indep. (modulo R^ω-embedability).
- $\chi_{orb}(\mathbb{X}_1, \dots, \mathbb{X}_n)$ depends only on the $W^*(\mathbb{X}_i)$'s.
- $\chi(X_1, \dots, X_n) = \chi_{orb}(X_1, \dots, X_n) + \sum_{i=1}^n \chi(X_i)$.

  ↑ ↑ single r.v.s → general case : open !

- a few computations : 2-proj.s = counterpart of single r.v. $\chi(X)$.
- unification : $\eta^* = -\chi_{orb}$, a major problem !

  ($\eta^* \leq -\chi_{orb}$ still very difficult !)

Coordinates : (Counterpart of $\mathbb{R}^n$)

- $C_R(\mathfrak{X}_i)$ = the univesal C^*-alg. gen. by $\mathfrak{X}_i = (x_{ij})_{j=1}^{l_i}$

  w/ $\|x_{ij}\| \le R$.

- $\mathfrak{X} = \mathfrak{X}_1 \sqcup \cdots \sqcup \mathfrak{X}_n$;

$$C_R(\mathfrak{X}) = \mathop{\ast}_{i=1}^{n} C_R(\mathfrak{X}_i).$$

- $A_i = (A_{ij})_{j=1}^{l_i}$ w/ $A_{ij} = A_{ij}^*$, $\|A_{ij}\| \le R$

  $\rightsquigarrow$ $*$-hom. $h \in C_R(\mathfrak{X}) \longmapsto h(A_i)$

  sending x_{ij} to A_{ij}.

Orbital free pressure ft:

$h = h^* \in C_R(\mathcal{X})$, $\tau_i \in TS(C_R(\mathcal{X}_i))$;

$$\pi_{orb,R}(h : (\tau_i)_{i=1}^n) := \lim_{\substack{m\to\infty \\ \delta\downarrow 0}} \overline{\lim_{N\to\infty}} \text{ of}$$

$$\sup_{\substack{A_i \in \\ \Gamma_R(\tau_i; N, m, \delta) \\ (1\le i\le n)}} \frac{1}{N^2} \log \int_{U(N)^n} d\gamma_N^{\otimes n} \exp\left(-N^2 \mathrm{tr}_N(h(V_i A_i V_i^*))\right)$$

(integral wrt the V_i's)

↑ can be replaced with "inf" if $\pi_{\tau_i}(C_R(\mathcal{X}_i))'' = HF$.

<u>Legendre transf.</u> : $\quad \tau \in TS(C_R(\mathbb{X}))$;

$$\eta_{orb,R}(\tau : (\tau_i)_{i=1}^{n})$$

$$= \inf \{ \tau(h) + \pi_{orb,R}(h : (\tau_i)_{i=1}^{n}) \mid h = h^* \in C_R(\mathbb{X}) \}.$$

"essential domain"

$$TS(C_R(\mathbb{X}) : (\tau_i)_{i=1}^{n}) = \{ \tau \in TS(C_R(\mathbb{X})) \mid \tau \restriction_{C_R(x_i)} = \tau_i \}.$$

$\mathbb{X} = \mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_n$ given in (M, τ)

$\longrightarrow \quad \tau_{\mathbb{X}_i} \in TS(C_R(x_i))\,, \quad \tau_{\mathbb{X}} \in TS(C_R(x))$

w/ $\|X_{ij}\| \leq R$

<u>Orbital</u> η-<u>entropy</u> :

$$\eta_{orb}(\mathbb{X}_1, \dots, \mathbb{X}_n) := \eta_{orb,R}(\tau_{\mathbb{X}} : (\tau_{\mathbb{X}_i})_{i=1}^n).$$

※ This is independent of the choice of cutoff const. $R > 0$.

Properties:

- $\eta_{orb} = 0 \longleftrightarrow$ free indep. (modulo R^ω-embedability).
- upper semicontinuous.
- $\mathcal{Y}_i \subseteq W^*(\mathcal{X}_i) \longrightarrow \eta_{orb}(\mathcal{X}_1, \dots, \mathcal{X}_n) \leq \eta_{orb}(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$.

  Hence $\eta_{orb}(\mathcal{X}_1, \dots, \mathcal{X}_n)$ depends only on the $W^*(\mathcal{X}_i)$'s.
- $\chi_{orb} \leq \eta_{orb}$.

  There exists an example with $\chi_{orb} \lneq \eta_{orb}$.

DEFN: $\tau \in TS(C_R(x))$.

τ = orbital equilibrium tracial state
aso. w/ $h = h^* \in C_R(x)$

$\overset{\longleftrightarrow}{\text{defn}}$ $\eta_{orb,R}(\tau : (\tau_i)_{i=1}^n)$

$$= \tau(h) + \pi_{orb,R}(h : (\tau_i)_{i=1}^n)$$ (finite)

w/ $\tau_i := \tau \restriction_{C_R(x_i)}$, $1 \leq i \leq n$.

REM: Given $(\tau_i)_{i=1}^n \to$

$\{ h = h^* \mid \exists 1$ orbital equilibrium tracial state aso. w/ $h \}$ G_δ-set

Matrix models:

$$\boxed{H}(N) = \left(\boxed{H}_i(N)\right)_{i=1}^{n}, \quad \boxed{H}_i(N) = \left(\xi_{ij}(N)\right)_{j=1}^{l_i}$$

$\xi_{ij}(N)$: $N \times N$ self-adj.s, $\|\cdot\| \leq R$.

$$\longrightarrow \tau_{\boxed{H}_i(N)} \in TS(C_R(x_i));$$

$$\tau_{\boxed{H}_i(N)}(x_{ij_1} \cdots x_{ij_r}) = tr_N(\xi_{ij_1}(N) \cdots \xi_{ij_r}(N)).$$

Assume:

$$\tau_{\boxed{H}_i(N)} \xrightarrow[N \to \infty]{w^*} \tau_i \in TS(C_R(x_i)).$$

$$\tau_i = HF \overset{\text{defn}}{\longleftrightarrow} \pi_{\tau_i}(C_R(x_i))'' = HF.$$

$h = h^* \in C_R(\mathcal{X})$ given.

- "orbital Gibbs microemsemble" $\mu_N^{(h,\Xi(N))}$:

$$\frac{1}{Z_N^{(h,\Xi(N))}} \exp\left(-N^2 \mathrm{tr}_N\left(h(V_i \Xi_i(N) V_i^*)\right)\right) d\gamma_N^{\otimes n}(V_i).$$

- "orbital mean tracial state" $\tau_N^{(h,\Xi(N))} \in TS(C_R(\mathcal{X}))$:

$$f \in C_R(\mathcal{X}) \longmapsto$$

$$\int_{U(N)^n} d\mu_N^{(h,\Xi(N))} \mathrm{tr}_N\left(f(V_i \Xi_i(N) V_i^*)\right) \in \mathbb{C}.$$

Proposition: Assume every τ_i = HF.

If $\tau \in TS(C_R(x))$ satisfies

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{(\varphi, \boxed{H}(N))}\left(\Gamma_{orb}(\tau : \boxed{H}(N) ; N, m, \delta)\right) = 0 \quad \text{—}(*)$$

for $^\forall$ large m, $^\forall$ small δ, then:

$$\begin{cases} \tau = \text{orbital equilibrium ass. w/ } \varphi, \\ \chi_{orb}(\tau) = \eta_{orb,R}(\tau). \end{cases}$$

Remark: $(*)$ ⟵ the almost sure convergence of the matrix model.

(Yes for "small" φ ; Collins, Guionnet, Segala)

Hiai's previous work:

$h = h^* \in C_R(\mathfrak{X})$

$\longrightarrow \quad \pi_R(h)$ free pressure ft

$\tau \in TS(C_R(\mathfrak{X}))$

$\longrightarrow \eta_R(\tau) = \inf\limits_{h=h^*} \tau(h) + \pi_R(h)$ Legendre transf.

- $\chi(\tau) \leq \eta_R(\tau)$.
- τ = equilibrium ass. w/ h $\overset{\text{defn}}{\longleftrightarrow}$ $\eta_R(\tau) = \tau(h) + \pi_R(h)$.

Let's assume : every $\mathfrak{X}_i$ = single x_i in what follows.

$\to \forall \tau_i \in TS(C_R(x_i) = C[-R,R])$, $\eta_R(\tau_i) = \chi(\tau_i)$.

$h = h^* \in C_R(\mathfrak{X})$ gives :

- "Gibbs microensemble" $\lambda_{R,N}^{h}$ on $((M_N^{sa})_{\underline{\underline{R}}})^n$:

$$\frac{1}{Z_{R,N}^{h}} \exp(-N^2 tr_N(h(A))) \, d\,Leb(A),$$

- "mean tracial state" $\tau_{R,N}^{h} \in TS(C_R(\mathfrak{X}))$:

$$f \in C_R(\mathfrak{X}) \mapsto \int d\lambda_{R,N}^{h} \, tr_N(f(A)) \in \mathbb{C}.$$

Proposition:

$$\pi_R(h) \geq \pi_{orb,R}(h : (z_i)_{i=1}^n) + \sum_{i=1}^n \chi(z_i).$$

"=" $\underset{iff}{\Longleftrightarrow}$ $^\forall$ large m, $^\forall$ small $\delta > 0$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \lambda_{R,N}^h \left(\prod_{i=1}^n \Gamma_R(z_i ; N, m, \delta) \right) = 0.$$

Cor:

$$\eta_R(z) \geq \eta_{orb,R}(z) + \sum_{i=1}^n \chi(z_i) \geq \chi_{orb}(z) + \sum_{i=1}^n \chi(z_i) = \chi(z).$$

If $\eta_R(z) = \chi(z)$ & every $\chi(z_i)$ finite, then:

$\eta_{orb,R}(z) = \chi_{orb}(z)$ holds.

Cor:

If z = equilibrium ass. w/ $\hat{h}$ & $\eta_R(z) = \chi(z)$,

then:

z = orbital equilibrium ass. w/ $\hat{h}$,

$$\pi_R(\hat{h}) = \pi_{orb,R}(z : (z_i)_{i=1}^{n}) + \sum_{i=1}^{n} \chi(z_i),$$

$$\eta_{orb,R}(z) = \chi_{orb}(z).$$

Proposition: Assume:

$$\lim_{N\to\infty} \frac{1}{N^2} \log \lambda^{h}_{R,N}(\Gamma_R(\tau; N, m, \delta)) = 0 \qquad -(**)$$

for $\forall$ large m, $\forall$ small $\delta > 0$.

Then

τ = equilibrium ass. w/ h & $\eta_R(\tau) = \chi(\tau)$.

(→ the previous Cor. Caldo !)

Remark: $(**) \leftarrow$ the almost sure convergence of the matrix model $\lambda^{h}_{R,N}$.

(Yes for "small" h ; Guionnet, Segala.)

Remark :

$$\tau_{R,N}^{h} \xrightarrow{w^*} \tau = \text{extremal}$$

$\rightsquigarrow$ (**) holds.

Biane – Dabrowski's concentration lemma

Q. Is the proposition still true for extremal <u>limit pts</u> ?

(the $\overline{\lim}$ is a trouble.)

Fact (following Biane-Dabrowski's argument):

$$K := \{ \sigma \in TS(C_R(\mathfrak{X})) \mid \sigma\restriction_{C_R(\mathfrak{X}_i)} = \tau_i \}$$

If a given $\tau \in K$ is weak*-exposed in K, then $\forall \gamma > 0$, $\exists\, p = p^* \in \mathbb{C}\langle \mathfrak{X} \rangle$;

$\forall$ prob. meas. μ on $U(N)^n$, $\forall m$, $\forall \delta > 0$,

$$\mu\left(\Gamma_{orb}(\tau : \boxed{H}(N) ; N, m, \delta)\right) > 1 - \gamma$$

as long as $\left| \tau_N^{(\mu, \boxed{H}(N))}(p) - \tau(p) \right| < \gamma/2$.

$$\tau_N^{(\mu,\Xi(N))}(f) := \int_{U(N)^n} d\mu(V_i)\, \mathrm{tr}_N\big(f(V_i\,\Xi_i(N)\,V_i^*)\big).$$

Q. Find an easy-to-use condition for a given $\tau \in K$, under which τ is weak*-exposed in K.

$$\boxed{\begin{array}{ccc} \chi & \longrightarrow & TS \\ & & \cup\!\shortparallel \\ \chi_{orb}/\eta_{orb} & \longrightarrow & K. \end{array}}$$