Edrei-Voiculescu Theorem

A "classical" approach

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Confused about Career Path
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Question: What do you see when you look at $\mathbb{R}^2$?

First Answer: A Hilbert space.

Diagnosis: A career in <u>functional analysis</u> may be right for you.

Second Answer: A symplectic manifold.

Diagnosis: A career in <u>symplectic geometry</u> may be right for you.

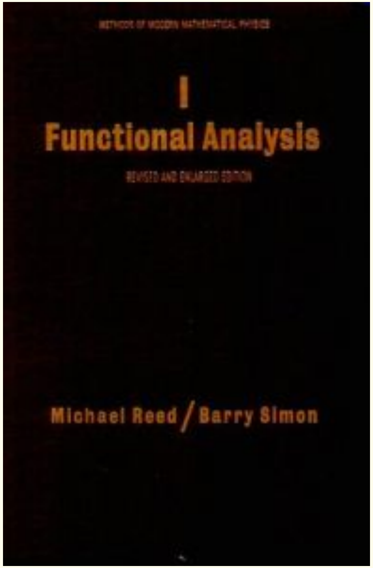
Question: What do positive definite functions mean to you?

First Answer: Traces.

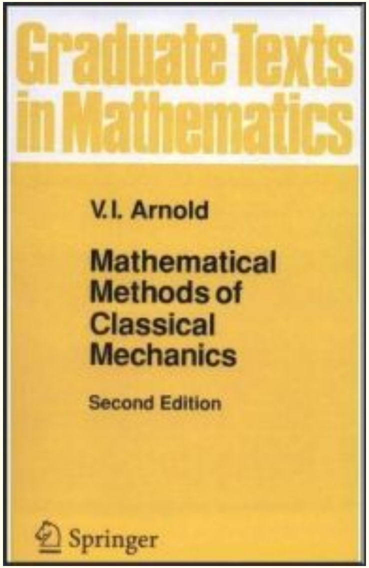
Diagnosis: A career in <u>quantum mechanics</u> may be right for you.

Second Answer: Fourier transforms.

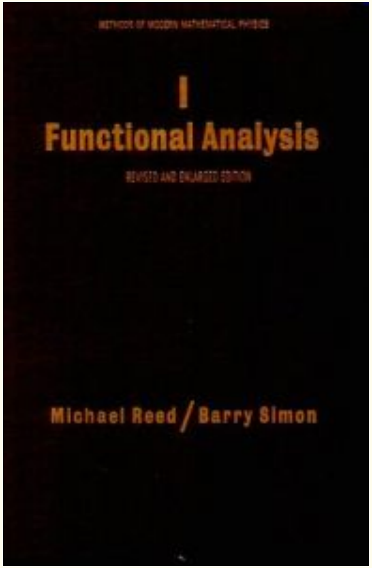
Diagnosis: A career in <u>classical mechanics</u> may be right for you.

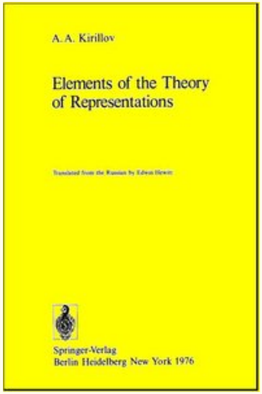


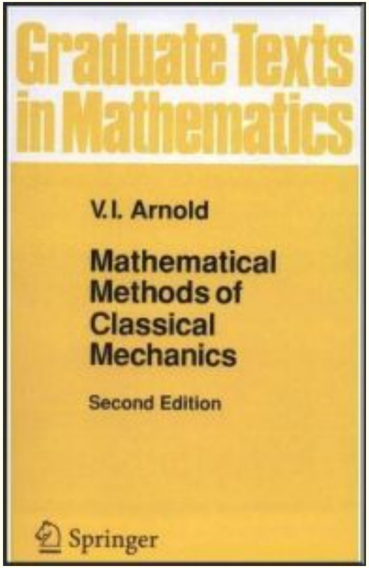
Hilbert Spaces



Symplectic Manifolds







Hilbert Spaces

Symplectic Manifolds

Kirillov's Philosophy

Let G be a Lie group.

- The extreme characters of G are parameterized by its "coadjoint orbits", each of which is a symplectic manifold (classical phase space).

- For $\mathcal{O} \subseteq \mathfrak{g}^*$ a coadjoint orbit, corresponding character is Fourier transform:

$$\Upsilon^{\mathcal{O}}(e^X) = \frac{1}{\mathrm{Vol}\,\mathcal{O}} \int_{\mathcal{O}} e^{\mathbf{i}\langle Y, X\rangle} dY.$$

- There is a correspondence $\mathcal{O} \longleftrightarrow (V, \rho)$ between coadjoint orbits and unirreps (quantum phase spaces) of G such that

$$\frac{\mathrm{Tr}\,\rho(e^X)}{\dim V} = \Upsilon^{\mathcal{O}}(e^X).$$

Kirillov's Philosophy

- The passage $\mathcal{O} \longrightarrow V$ is a mathematically meaningful formalization of "quantization."

- The correspondence $\mathcal{O} \longleftrightarrow V$ works perfectly for nilpotent groups.

- The correspondence $\mathcal{O} \longleftrightarrow V$ is sometimes false, the Kirillov character formula is sometimes wrong, but typically these issues can be patched.

Kirillov's Philosophy for $U(N)$

- Coadjoint representation amounts to action on $H(N)$ by conjugation.

- Each orbit $\mathcal{O}$ has a "signature" $y=(y_1,\dots,y_N)$, $y_1\geq\dots\geq y_N$.

- Only <u>integral</u> coadjoint orbits yield extreme characters: $\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{Z}^N$.

- Kirillov's character formula:

$$\Upsilon^\lambda(e^{\mathbf{i}X})=\frac{1}{\mathcal{J}(X)}\int_{\mathcal{O}_{\lambda+\rho}} e^{\mathbf{i}\,\mathrm{Tr}(XY)}\,dY \qquad \left\{\begin{array}{l}\rho=(N-1,\dots,2,1,0)\\ \mathcal{J}=\text{Jacobian of exp.}\end{array}\right.$$

THEOREM (Bott, 1954): For $d \geq 2$ and $N \geq \frac{d+1}{2}$, $\pi_d U(N) = \begin{cases} 0, & \text{if } d \text{ even} \\ \mathbb{Z}, & \text{if } d \text{ odd.} \end{cases}$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
$U(1)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
$U(2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
$U(3)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	?	?	?	?
$U(4)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	?	?
$U(5)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?

Unstable Range

Stable Range

COROLLARY (Bott Periodicity): The infinite-dimensional (or "stable") unitary group $U(\infty) = \varinjlim U(N)$ has homotopy sequence

$$0, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \ldots$$

Theorem (Voiculescu, 1978): The extreme characters of $U(\infty)$ are parameterized by the points of

$$\mathcal{E} = \left\{ \left(\alpha_1^{\pm} \geq \alpha_2^{\pm} \geq \ldots \geq 0;\ \beta_1^{\pm} \geq \beta_2^{\pm} \geq \ldots \geq 0;\ \delta^{\pm} \right) : \sum_{i=1}^{\infty} \left(\alpha_i^{\pm} + \beta_i^{\pm} \right) \leq \delta^{\pm} \right\}.$$

For each point $(\alpha^{\pm}; \beta^{\pm}; \delta^{\pm}) \in \mathcal{E}$, the corresponding character is

$$\Upsilon^{(\alpha^{\pm}; \beta^{\pm}; \delta^{\pm})}\left(e^{\mathbf{i}x_1}, \ldots, e^{\mathbf{i}x_k}, 1, 1, \ldots\right) = E^{(\alpha^{\pm}; \beta^{\pm}; \delta^{\pm})}\left(e^{\mathbf{i}x_1}\right) \ldots E^{(\alpha^{\pm}; \beta^{\pm}; \delta^{\pm})}\left(e^{\mathbf{i}x_k}\right) \qquad (!!)$$

where

$$E^{(\alpha^{\pm}; \beta^{\pm}; \delta^{\pm})}(z) = e^{\gamma^{+}(z-1) + \gamma^{-}(z^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^{+}(z-1)}{1-\alpha_i^{+}(z-1)} \frac{1+\beta_i^{-}(z^{-1}-1)}{1-\alpha_i^{-}(z^{-1}-1)}$$

is the "Edrei function," and

$$\gamma^{\pm} = \delta^{\pm} - \sum_{i=1}^{\infty} \left(\alpha_i^{\pm} + \beta_i^{\pm} \right)$$

is the "defect" parameter.

Definition (Schoenberg, 1948): A function $\pi:\mathbb{Z}\to\mathbb{R}$ is <u>totally positive</u> if the Toeplitz matrix $[\pi(j-i)]_{i,j\in\mathbb{Z}}$ has no negative minors.

$$\begin{matrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \pi(-1) & \pi(0) & \pi(1) & \pi(2) & \pi(3) & \cdots \\ \cdots & \pi(-2) & \pi(-1) & \pi(0) & \pi(1) & \pi(2) & \cdots \\ \cdots & \pi(-3) & \pi(-2) & \pi(-1) & \pi(0) & \pi(1) & \cdots \\ \cdots & \pi(-4) & \pi(-3) & \pi(-2) & \pi(-1) & \pi(0) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Theorem (Edrei, 1952): π is TP iff $\sum_{k\in\mathbb{Z}}\pi(k)z^k = E^{(\alpha^\pm;\beta^\pm;\delta^\pm)}(z)$ for some $(\alpha^\pm;\beta^\pm;\delta^\pm)\in\mathcal{E}$,

$$\ldots+\pi(-2)z^{-2}+\pi(-1)z^{-1}+\pi(0)+\pi(1)z^{1}+\pi(2)z^{2}+\ldots = \text{an Edrei function.}$$

Theorem (Vershik – Kerov, 1982): Every extreme character Υ of $U(\infty)$ is the large N limit of a sequence of extreme $U(N)$ characters. That is, there exists a sequence

$$\lambda^{(N)} = (\lambda_1^{(N)}, \dots, \lambda_N^{(N)}), \quad N \geq 1,$$

of integral signatures such that

$$\Upsilon = \lim_{N \to \infty} \Upsilon^{\lambda^{(N)}}$$

uniformly on compact subsets of the infinite torus

$$U(1) \times U(1) \times U(1) \times \dots$$

- Practical consequence: can recover Edrei-Voiculescu if we can classify sequences $\lambda^{(N)} = (\lambda_1^{(N)}, \ldots, \lambda_N^{(N)})$ of integral signatures such that the limits

$$\lim_{N\to\infty} T^{\lambda^{(N)}}(\underbrace{e^{\mathbf{i}x_1}, \ldots, e^{\mathbf{i}x_k}}_{k}, \underbrace{1, 1, \ldots 1}_{N-k})$$

exist, and compute the corresponding limits.

Quantum option: Weyl character formula,

$$T^{\lambda^{(N)}}(e^{\mathbf{i}x_1}, \ldots e^{\mathbf{i}x_k}, 1, 1, \ldots 1) = \frac{S_{\lambda^{(N)}}(e^{\mathbf{i}x_1}, \ldots, e^{\mathbf{i}x_k}, 1, \ldots, 1)}{S_{\lambda^{(N)}}(1, 1, \ldots, 1)}.$$

Classical option: Kirillov character formula,

$$T^{\lambda^{(N)}}(e^{\mathbf{i}x_1}, \ldots e^{\mathbf{i}x_k}, 1, 1, \ldots 1) = \frac{1}{\mathrm{Vol}\, \mathcal{O}_{\lambda^{(N)}+\rho^{(N)}} \cdot \mathcal{J}(x_1, \ldots, x_k, 0, \ldots, 0)} \int_{\mathcal{O}_{\lambda^{(N)}+\rho^{(N)}}} e^{\mathbf{i}\,\mathrm{Tr}(X_k Y)}\, dY.$$

Char $U(\infty)$

GOAL

Reduction
(Voiculescu)

STRATEGY

Approximation
(Vershik-Kerov)

METHODS

Function Theory
(Edrei)

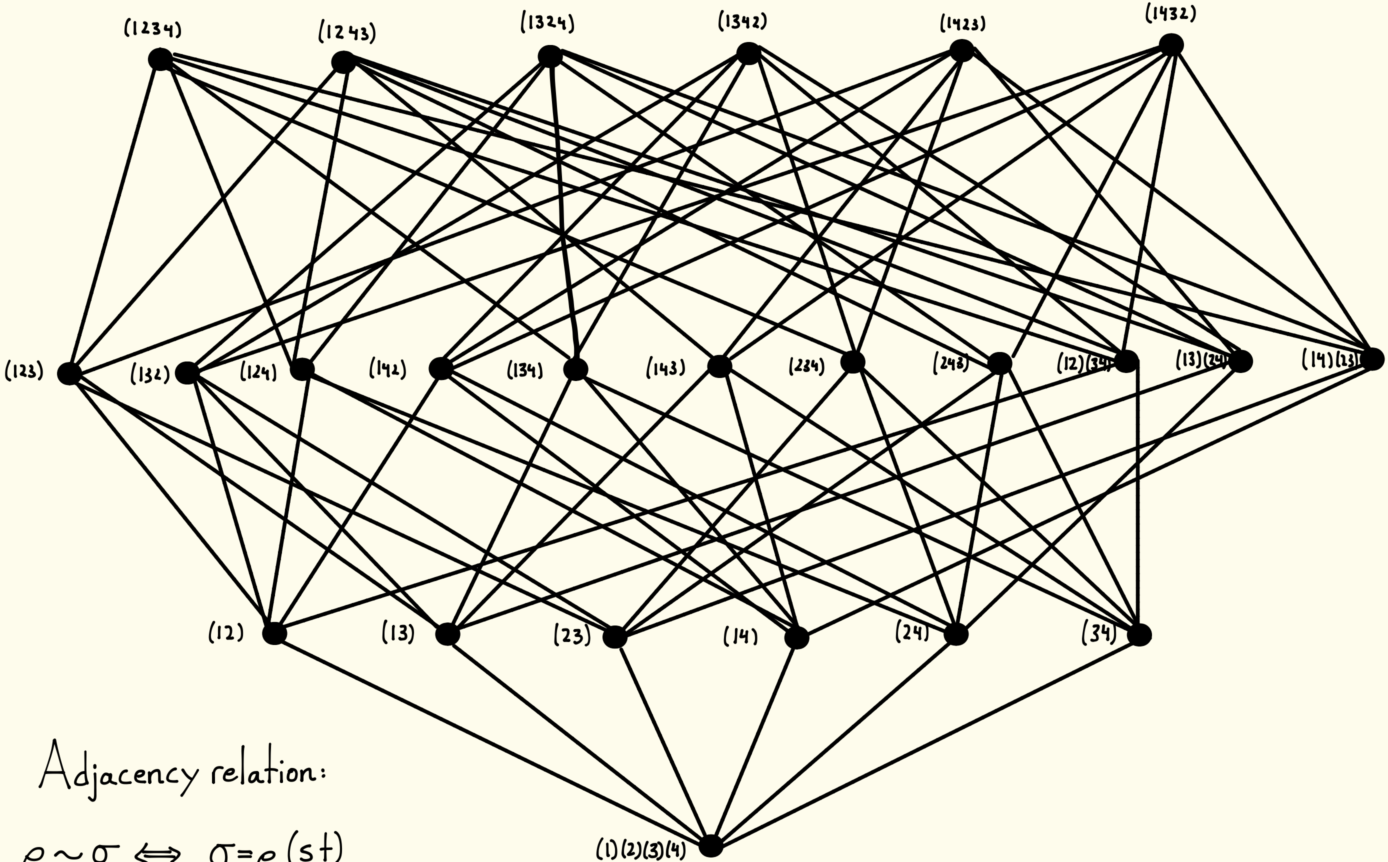
Weyl asymptotics
(Okounkov-Olshanski;
Borodin-Olshanski
Petrov
Gorin-Panova)

Kirillov asymptotics
(Unexplored)

Question: What is the combinatorics of type A representation theory on "the Kirillov side?"

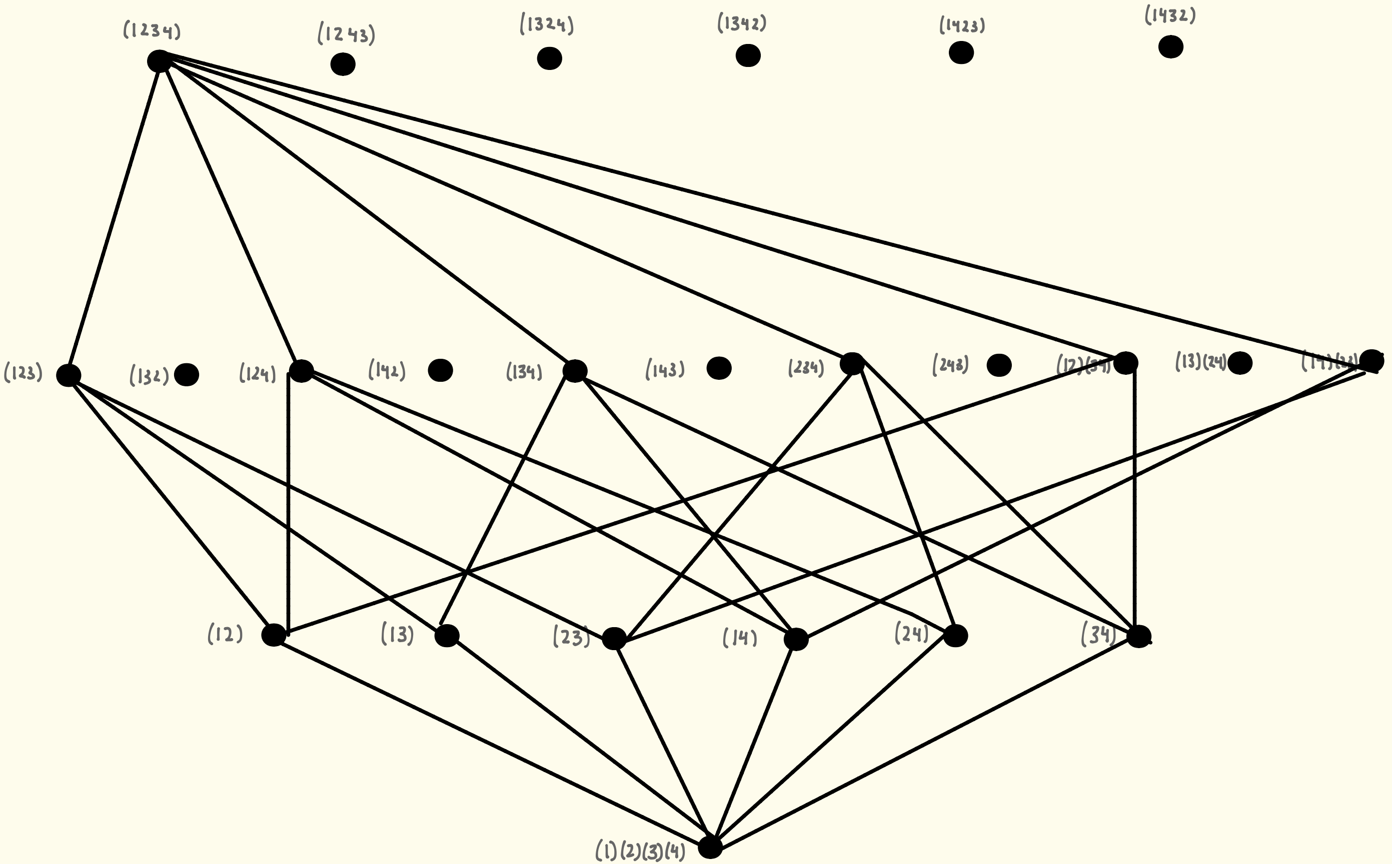
Pre-answer: It is natural to expect the combinatorics of permutations to appear. But, in what form?

Answer: The missing combinatorial apparatus is a desymmetrized version of Hurwitz Theory.



Adjacency relation:

$\rho \sim \sigma \iff \sigma = \rho(st)$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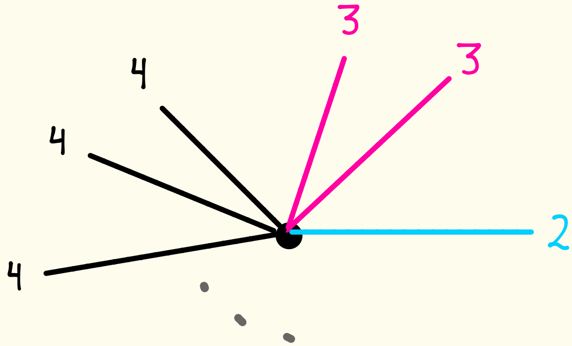
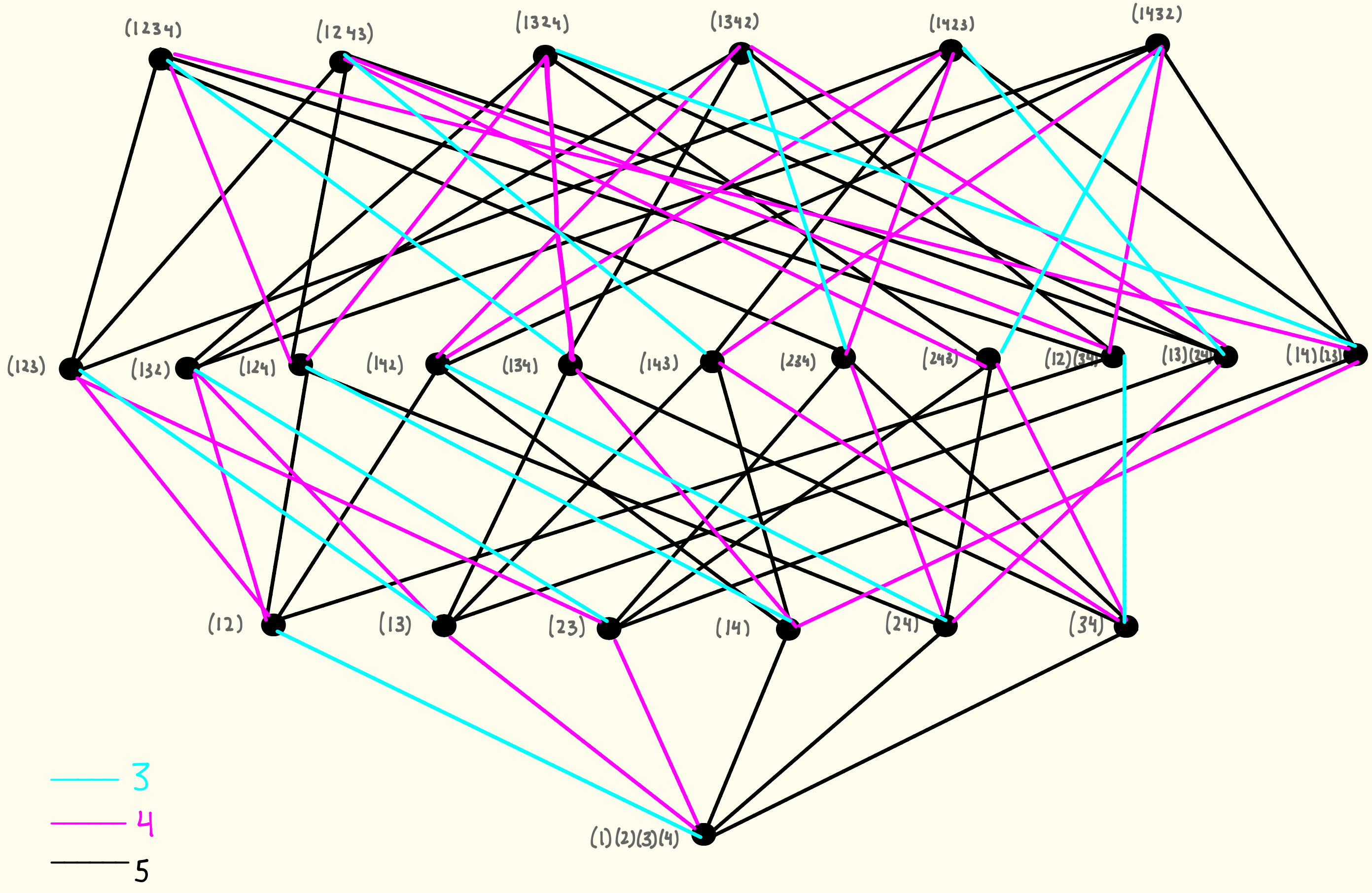
- A fragment of the Cayley graph of $S(d)$:

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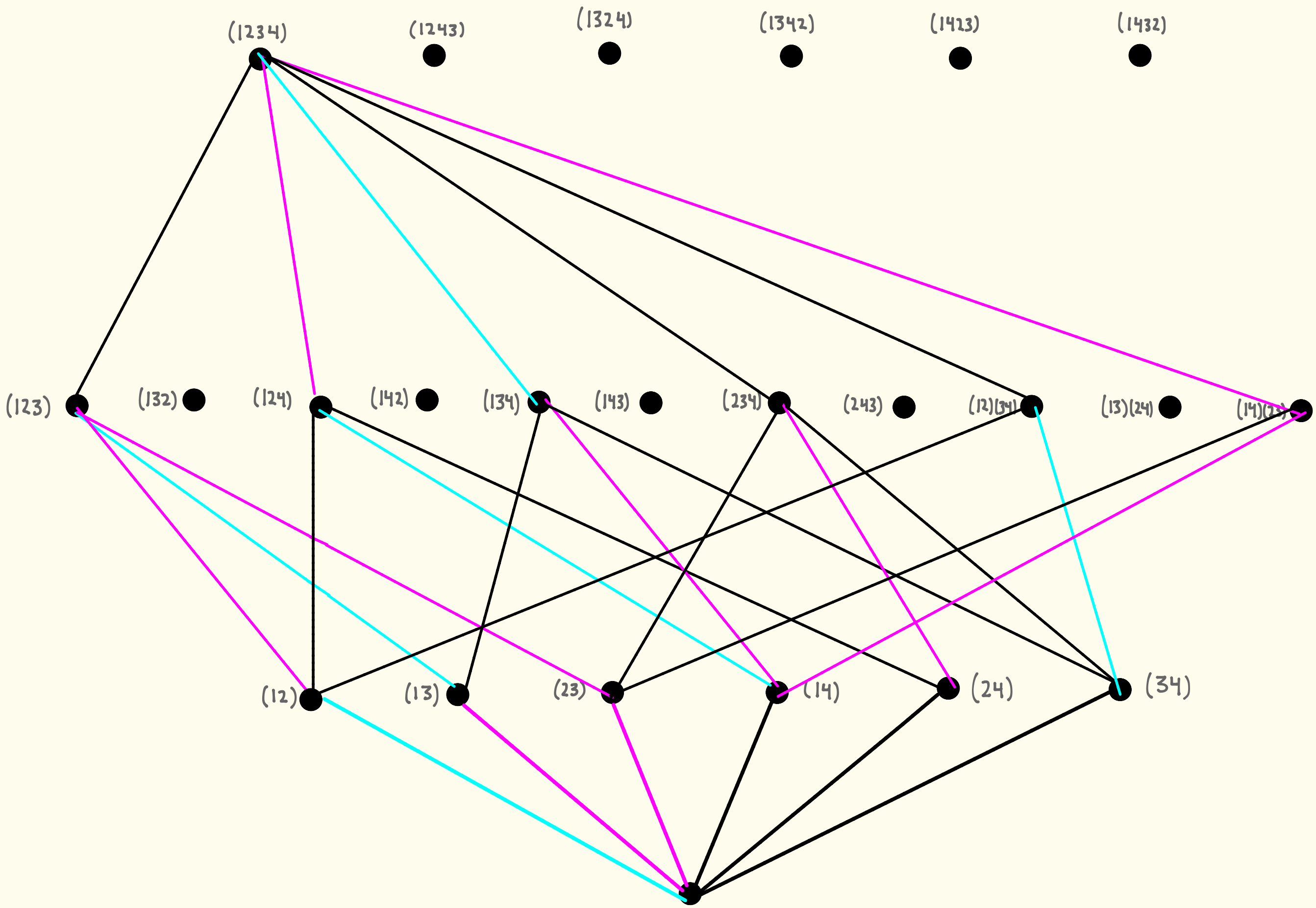
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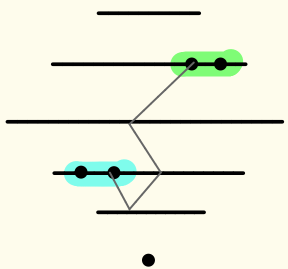
Definition (Okounkov, 2001): The <u>double Hurwitz number</u> $H^r(\mu,\nu)$ counts r-step walks on $S(d)$ beginning at a point of C_μ and ending at a point of C_ν such that the endpoints and steps together generate a transitive subgroup of $S(d)$. Note

$$H^r(\mu,\nu) \neq 0 \Leftrightarrow r = 2g-2+\ell(\mu)+\ell(\nu) \quad \text{for some } g \geq 0;$$

$$H^r(\mu,\nu) = H_g(\mu,\nu)$$

Definition (Goulden, Guay-Paquet, N., 2011): The <u>monotone</u> double Hurwitz number $\vec{H}^r(\mu,\nu)$ counts walks as above with the property that the labels of the edges traversed form a weakly increasing sequence.



Theorem (Goulden, Guay-Paquet, N., 2011): Consider the Laplace transform

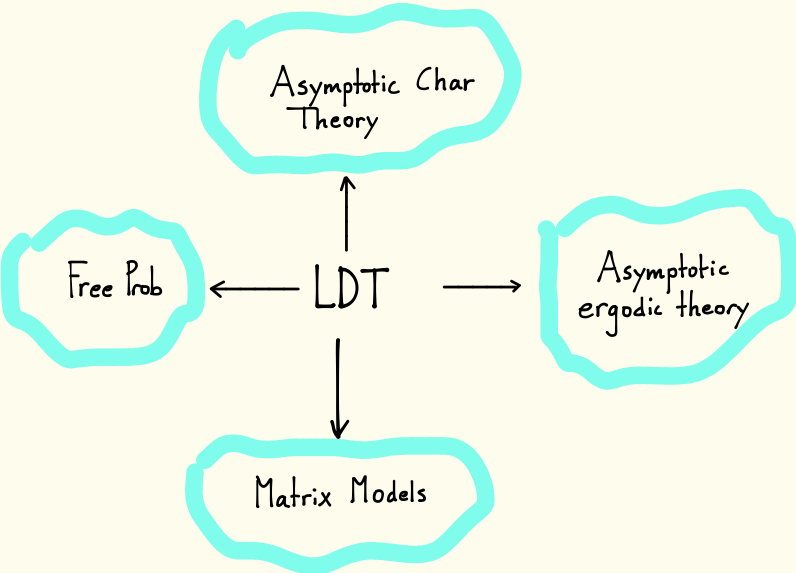
$$L(X;z) = \frac{1}{\mathrm{Vol}\,\mathcal{O}_y} \int_{\mathcal{O}_y} e^{z\,\mathrm{Tr}(XY)}\, dY$$

of normalized Liouville measure on the coadjoint orbit of $U(N)$ through $y=(y_1,\dots,y_N)$.
We have

$$\log L(X;z) = \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{g=0}^{\infty} \underbrace{N^{2-d} \sum_{\mu,\nu \vdash d} (-1)^{\ell(\mu)+\ell(\nu)} \vec{H}_g(\mu,\nu) \frac{p_\mu(x_1,\dots,x_N)}{N^{\ell(\mu)}} \frac{p_\nu(y_1,\dots,y_N)}{N^{\ell(\nu)}}}_{\text{"basic sums"}} + O(z^{N+1}).$$

Question: Can you get from the Leading Derivatives Theorem to the Edrei-Voiculescu theorem?

Answer: YES. Actually, you can get to a number of seemingly different islands:



- The underlying problem is: given two triangular arrays

$$
\begin{array}{llll}
x_1^{(1)} & & & \\
x_1^{(2)} & x_2^{(2)} & & \\
\vdots & \vdots & \ddots & \\
x_1^{(N)} & x_2^{(N)} & \cdots & x_N^{(N)} \quad \ddots \\
\vdots & \vdots & & \vdots
\end{array}
\qquad\qquad
\begin{array}{llll}
y_1^{(1)} & & & \\
y_1^{(2)} & y_2^{(2)} & & \\
\vdots & \vdots & \ddots & \\
y_1^{(N)} & y_2^{(N)} & \cdots & y_N^{(N)} \quad \ddots \\
\vdots & \vdots & & \vdots
\end{array}
$$

determine large N behaviour of the "basic sums"

$$
N^{2-d} \sum_{\mu,\nu \vdash d} (-1)^{\ell(\mu)+\ell(\nu)} \vec{H}_g(\mu,\nu) \frac{p_\mu(x_1^{(N)},\ldots,x_N^{(N)})}{N^{\ell(\mu)}} \frac{p_\nu(y_1^{(N)},\ldots,y_N^{(N)})}{N^{\ell(\nu)}} .
$$

$$\{x_1^{(N)},\dots,x_N^{(N)}\} = \{x_1,\dots,x_k\}$$

$$\{y_1^{(N)},\dots,y_N^{(N)}\} = \{\lambda_1^{(N)}+N-1,\dots,\lambda_N^{(N)}\}$$

$$\Psi_\nu = \lim_{N\to\infty} p_\nu\left(\frac{\lambda_1^{(N)}-1}{N},\dots,\frac{\lambda_N^{(N)}-N}{N}\right)$$

Asymptotic Char Theory

$$\uparrow$$

$$\{x_1^{(N)},\dots,x_N^{(N)}\} = \{x_1,\dots,x_k\}$$

Free Prob

$$\Psi_\nu = \lim_{N\to\infty} \frac{p_\nu\left(\frac{y_1^{(N)}}{N},\dots,\frac{y_N^{(N)}}{N}\right)}{N^{\ell(\nu)}}$$

$$\longleftarrow \quad N^{2-d}\sum_{\mu,\nu\vdash d}(-1)^{\ell(\mu)+\ell(\nu)}\vec{H}_g(\mu,\nu)\,\frac{p_\mu(x_1^{(N)},\dots,x_N^{(N)})}{N^{\ell(\mu)}}\,\frac{p_\nu(y_1^{(N)},\dots,y_N^{(N)})}{N^{\ell(\nu)}} \quad \longrightarrow$$

$$\{x_1^{(N)},\dots,x_N^{(N)}\} = \{x_1,\dots,x_k\}$$

Asymptotic ergodic theory

$$\Psi_\nu = \lim_{N\to\infty} p_\nu\left(\frac{y_1^{(N)}}{N},\dots,\frac{y_N^{(N)}}{N}\right)$$

$$\downarrow$$

Matrix Models

$$\phi_\mu = \lim_{N\to\infty} \frac{p_\mu\left(\frac{x_1^{(N)}}{\sqrt{N}},\dots,\frac{x_N^{(N)}}{\sqrt{N}}\right)}{N^{\ell(\mu)}} \qquad \Psi_\nu = \lim_{N\to\infty} \frac{p_\nu\left(\frac{y_N^{(N)}}{\sqrt{N}},\dots,\frac{y_N^{(N)}}{\sqrt{N}}\right)}{N^{\ell(\nu)}}$$

- Infinite Hermitian matrices: $H(\infty) = \varinjlim H(N)$, $\mathrm{H} = \varprojlim H(N)$.

- Bochner's thm holds: a Borel measure μ on H is characterized by

$$F_\mu(X) = \int_{\mathrm{H}} e^{i\langle X,Y\rangle} \mu(dY), \qquad X \in H(\infty).$$

- $U(\infty)$ acts on H by conjugation: $\rho(U)Y = UYU^{-1}$.

- We are interested in characterizing $U(\infty)$-invariant measures on H. They form a convex set, the extreme points of which are the ergodic measures.

Theorem (Olshanski, Vershik, 1996): Every ergodic $U(\infty)$-invariant measure on H is the large N limit of a $U(N)$-invariant ergodic measure on $H(N)$.

Practical Consequences:

- The $U(N)$-invariant measures on H are convex combinations of orbital measures.

- Just need to classify sequences $y^{(N)} = (y_1^{(N)}, \dots, y_N^{(N)})$ such that

$$\lim_{N\to\infty} \frac{1}{\mathrm{Vol}\, \mathcal{O}_{y^{(N)}}} \int_{\mathcal{O}_{y^{(N)}}} e^{i\,\mathrm{Tr}(X_k Y)}\, dY, \qquad X_k = \mathrm{diag}(x_1, \dots, x_k, 0, 0, \dots) \in H(\infty)$$

exists, and compute these limits.

- The function
$$\log \frac{1}{\mathrm{Vol}\, \mathcal{O}_{y^{(N)}}} \int_{\mathcal{O}_{y^{(N)}}} e^{z \operatorname{Tr}(X_k Y)} \, dY$$
is a power series in z, with coefficient of $\frac{z^d}{d!}$ being

$$\sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{\mu,\nu \vdash d} (-1)^{\ell(\mu)+\ell(\nu)} \vec{H}_g(\mu,\nu) \frac{p_\mu(x_1,\dots,x_k)}{N^{\ell(\mu)+\ell(\nu)-2}} \, p_\nu\left(\frac{y_1^{(N)}}{N}, \dots, \frac{y_N^{(N)}}{N}\right) + O\left(\frac{1}{N^2}\right).$$

- For example,

$$\vec{H}_g(3,3)\, p_3(x)\, p_3\left(\frac{y_i^{(N)}}{N}\right) - \vec{H}_g(3,21) \frac{p_3(x)}{N} p_{21}\left(\frac{y_i^{(N)}}{N}\right) + \vec{H}_g(3,111) \frac{p_3(x)}{N^2} p_{111}\left(\frac{y_i^{(N)}}{N}\right)$$

$$- \vec{H}_g(21,3) \frac{p_{21}(x)}{N} p_3\left(\frac{y_i^{(N)}}{N}\right) + \vec{H}_g(21,21) \frac{p_{21}(x)}{N^2} p_{21}\left(\frac{y_i^{(N)}}{N}\right) - \vec{H}_g(21,21) \frac{p_{21}(x)}{N^3} p_{111}\left(\frac{y_i^{(N)}}{N}\right)$$

$$+ \vec{H}_g(111,3) \frac{p_{21}(x)}{N^2} p_3\left(\frac{y_i^{(N)}}{N}\right) - \vec{H}_g(111,21) \frac{p_{21}(x)}{N^3} p_{21}\left(\frac{y_i^{(N)}}{N}\right) + \vec{H}_g(111,111) \frac{p_{21}(x)}{N^4} p_{111}\left(\frac{y_i^{(N)}}{N}\right)$$

- Suppose $\psi_\nu := \lim_{N\to\infty} p_\nu\left(\frac{y_1^{(N)}}{N}, \dots, \frac{y_N^{(N)}}{N}\right)$ exists for each $\nu \vdash d$. Then,

$$\lim_{N\to\infty} \sum_{\mu,\nu\vdash d} (-1)^{\ell(\mu)+\ell(\nu)} \vec{H}_g(\mu,\nu) \frac{p_\mu(x_1,\dots,x_k)}{N^{\ell(\mu)+\ell(\nu)-2}} \, p_\nu\left(\frac{y_1^{(N)}}{N},\dots,\frac{y_N^{(N)}}{N}\right) = \vec{H}_g(d,d)\, p_d(x_1,\dots,x_k)\,\psi_\nu$$

$$= \vec{H}_g(d,d)\, x_1^d\, \psi_\nu + \dots + \vec{H}_g(d,d)\, x_k^d\, \psi_\nu. \qquad (\text{additivity}\,!!).$$

- In genus zero, $\vec{H}_0(d,d) = (d-1)!$ is just Card $C_{(d)}$.

- As a function germ,

$$\lim_{N\to\infty} \log \frac{1}{\mathrm{Vol}\, \mathcal{O}_{y^{(N)}}} \int_{\mathcal{O}_{y^{(N)}}} e^{z\,\mathrm{Tr}(X_k Y)}\, dY = \sum_{d=1}^{\infty} \psi_d \frac{(z x_1)^d}{d} + \dots + \sum_{d=1}^{\infty} \psi_d \frac{(z x_k)^d}{d}.$$

Proposition (Olshanski, Vershik): The limit

$$\psi_r = \lim_{N\to\infty} p_r\left(\frac{y_1^{(N)}}{N}, \dots, \frac{y_N^{(N)}}{N}\right)$$

exists for any fixed r iff the limits ψ_1, ψ_2 and $\alpha_i = \lim_{N\to\infty} \frac{y_i^{(N)}}{N}$ exist.

In this case, $p_2(\alpha_1, \alpha_2, \dots) \leq \psi_2$, and $\psi_d = p_d(\alpha_1, \alpha_2, \dots)$ for $d \geq 3$.

- This defines the "Olshanski-Vershik specialization"

$$p_1 \mapsto \psi_1$$

$$p_2 \mapsto \gamma + p_2(\alpha_1, \alpha_2, \dots), \qquad \gamma = \psi_2 - p_2(\alpha_1, \alpha_2, \dots)$$

$$p_3 \mapsto p_3(\alpha_1, \alpha_2, \dots)$$

$$p_4 \mapsto p_4(\alpha_1, \alpha_2, \dots)$$

$$\vdots$$

- Returning to the formula

$$\lim_{N\to\infty} \log \frac{1}{\mathrm{Vol}\, \mathcal{O}_{y^{(N)}}} \int_{\mathcal{O}_{y^{(N)}}} e^{z \operatorname{Tr}(X_k Y)}\, dY = \sum_{d=1}^{\infty} \psi_d \frac{(zx_1)^d}{d} + \dots + \sum_{d=1}^{\infty} \psi_d \frac{(zx_k)^d}{d},$$

We can now compute the sums on the RHS as follows:

$$\sum_{d=1}^{\infty} \psi_d \frac{(zx_j)^d}{d} = \psi_1 \frac{zx_j}{1} + \psi_2 \frac{(zx_j)^2}{2} + \sum_{d=3}^{\infty} p_d(\alpha_1, \alpha_2, \dots) \frac{(zx_j)^d}{d}$$

$$= \psi_1 \frac{zx_j}{1} + \gamma \frac{(zx_j)^2}{2} - p_1(\alpha) \frac{zx_j}{1} + \sum_{d=1}^{\infty} p_d(\alpha) \frac{(zx_j)^d}{d}$$

- Exponentiating yields

$$e^{\sum_{d=1}^{\infty} \psi_d \frac{(zx_j)^d}{d}} = e^{\psi_1 z x_j + \gamma \frac{(zx_j)^2}{2}} \, e^{-p_1(\alpha) z x_j} \, e^{\sum_{d=1}^{\infty} p_d(\alpha) \frac{(zx_j)^d}{d}}$$

$$= e^{\psi_1 z x_j + \gamma \frac{(zx_j)^2}{2}} \prod_{i=1}^{\infty} e^{-\alpha_i z x_j} \sum_{d=0}^{\infty} h_d(\alpha) \, (zx_j)^d$$

$$= e^{\psi_1 z x_j + \gamma \frac{(zx_j)^2}{2}} \prod_{i=1}^{\infty} \frac{e^{-\alpha_i z x_j}}{1 - \alpha_i z x_j} .$$

- Conclusion:

$$\lim_{N \to \infty} \frac{1}{\mathrm{Vol}\, \mathcal{O}_{y^{(N)}}} \int_{\mathcal{O}_{y^{(N)}}} e^{i \operatorname{Tr}(X_k Y)} \, dY = \prod_{j=1}^{k} e^{i\psi_1 x_j - \gamma \frac{x_j^2}{2}} \prod_{i=1}^{\infty} \frac{e^{-i\alpha_i x_j}}{1 - i\alpha_i x_j} .$$

Theorem (Vershik, Kerov): The ergodic $U(\infty)$-invariant measures on H are parameterized by the points of

$$\mathcal{P} = \left\{ (\alpha_1 \geq \alpha_2 \geq \ldots ; \psi_1, \psi_2) : \sum_{i=1}^{\infty} \alpha_i^2 \leq \psi_2 \right\}.$$

The Fourier transform of the ergodic measure $\mu^{(\alpha; \psi_1, \psi_2)}$ is

$$F_{\mu^{(\alpha;\psi_1,\psi_2)}}(x_1, x_2, \ldots, x_k, 0, 0, \ldots) = P^{(\alpha;\psi_1,\psi_2)}(x_1) \ldots P^{(\alpha;\psi_1,\psi_2)}(x_k),$$

where $P^{(\alpha;\psi_1,\psi_2)}(x)$ is the "Polya function"

$$P^{(\alpha;\psi_1,\psi_2)}(x) = e^{i\psi_1 x - \gamma \frac{x^2}{2}} \prod_{i=1}^{\infty} \frac{e^{-i\alpha_i x}}{1 - i\alpha_i x}$$

and $\gamma = \psi_2 - \sum_{i=1}^{\infty} \alpha_i^2$ is the "defect."

Definition (Schoenberg): A function $\pi:\mathbb{R}\to\mathbb{R}$ is <u>totally positive</u> if, for any two sequences $\dots s_{-2}<s_{-1}<s_0<s_1<s_2\dots$ and $\dots t_{-2}<t_{-1}<t_0<t_1<t_2<\dots$, the matrix $[\pi(t_j-s_i)]_{i,j\in\mathbb{Z}}$ has nonnegative minors:

$$\begin{matrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & \pi(t_{-2}-s_{-1}) & \pi(t_{-1}-s_{-1}) & \pi(t_0-s_{-1}) & \pi(t_1-s_{-1}) & \pi(t_2-s_{-1}) & \dots \\ \dots & \pi(t_{-2}-s_0) & \pi(t_{-1}-s_0) & \pi(t_0-s_0) & \pi(t_1-s_0) & \pi(t_2-s_0) & \dots \\ \dots & \pi(t_{-2}-s_1) & \pi(t_{-1}-s_1) & \pi(t_0-s_1) & \pi(t_1-s_1) & \pi(t_2-s_1) & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Theorem (Schoenberg, 1948): π is TP iff its Fourier transform is a Pólya function.

- The function

$$\log \frac{1}{\mathrm{Vol}\, \mathcal{O}_{\lambda^{(N)}+\rho^{(N)}}} \int_{\mathcal{O}_{\lambda^{(N)}+\rho^{(N)}}} e^{z \,\mathrm{Tr}(X_k Y)}\, dY$$

is a power series in z, with coefficient of $\frac{z^d}{d!}$ being

$$\sum_{g=0}^{\infty} \frac{1}{N^{2g}} \underbrace{\sum_{\mu,\nu \vdash d} (-1)^{\ell(\mu)+\ell(\nu)} \vec{H}_g(\mu,\nu) \frac{p_\mu(x_1,\dots,x_n)}{N^{\ell(\mu)+\ell(\nu)-2}}\, p_\nu\left(1+\frac{\lambda_1^{(N)}-1}{N}, \dots, 1+\frac{\lambda_N^{(N)}-N}{N}\right)}.$$

- The large N limit of the basic sums can be computed much as in the ergodic case, though the combinatorics of $\vec{H}_g(\mu,\nu)$ enters non-trivially (don't forget about g).

- Reduces to analogous calculation with $\psi_\nu = \lim_{N\to\infty} p_\nu\left(\frac{\lambda_1^{(N)}-1}{N}, \dots, \frac{\lambda^{(N)}-N}{N}\right)$ given by <u>Vershik-Kerov specialization</u>.