Quantized free convolution via representations of classical Lie groups.

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Overview

Classical real Lie groups

- unitary matrices $U(N)$
- orthogonal $SO(2N+1)$
- symplectic $Sp(2N)$
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All depend on an integral parameter N.

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For example, what about

A Restrictions to subgroups (e.g. $U(k) \subset U(N)$)
   - Of finite rank
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Semistandard Young tableaux

B Tensor products

Littlewood–Richardson coefficients

Irreducible representations and characters of $U(N)$

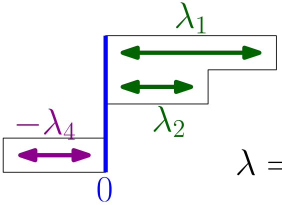
$U(N)$ — group of all $N \times N$ unitary matrices. T — representation of $U(N)$, i.e. homomorphism

$$T : U(N) \mapsto GL(V).$$

T is **irreducible** if V has no nontrivial $U(N)$–invariant subspaces.

Irreducible representations and characters of $U(N)$

Theorem. (E. Cartan, H. Weyl, 1920s) Irreducible representations of $U(N)$ are parameterized by N–tuples of integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.



(signatures)

$$\lambda = (5, 3, 0, -3)$$

The character of representation T_λ is given by

$$\chi_\lambda(U) = \mathrm{Trace}\, T_\lambda(U) = s_\lambda(u_1, \ldots, u_N) = \frac{\det\left(u_i^{\lambda_j+N-j}\right)_{i,j=1}^{N}}{\prod_{i<j}(u_i - u_j)},$$

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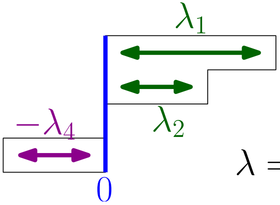

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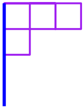
Remark. Very similar formulas exist for groups $Sp(2N)$ and $SO(N)$. All later results are also proved for these groups as well.

Different ways to grow $\lambda(N)$

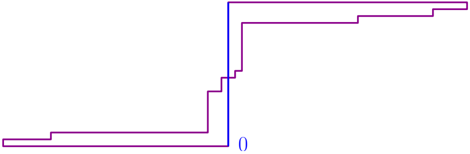
We want signature $\lambda = \lambda(N)$ to somehow *grow* as $N \to \infty$.

Different ways to grow $\lambda(N)$

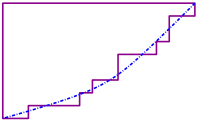
1. The rows stabilize except for the tail of 0s. (“finite” signatures)



2. *Some* of the rows grow linearly. (“thin” signatures)



3. All rows grow linearly in N. (“thick” signatures)



4. Rows grow superlinearly, i.e. $\lambda_i(N) \gg N$. (“very thick” signatures)

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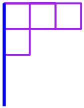
Degeneration into the symmetric group. (Schur–Weyl duality)



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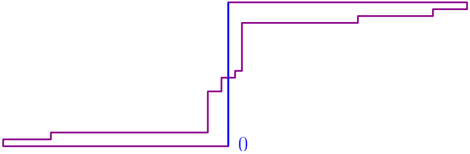

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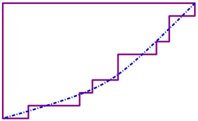

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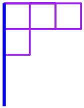

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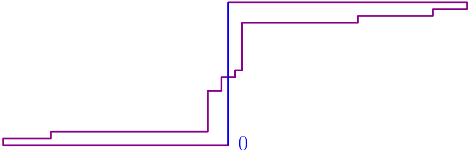
Representation theory of $U(\infty)$.
(Voiculescu, Vershik–Kerov, etc)



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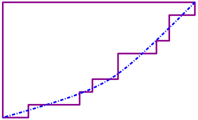

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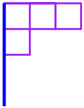

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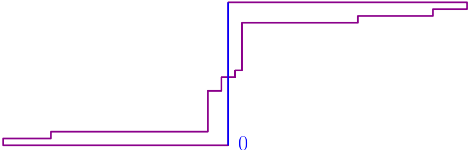

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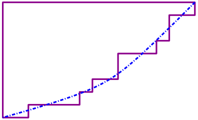

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"Semiclassical limit" to RMT.
(Biane, Collins–Sniady)

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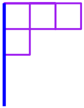

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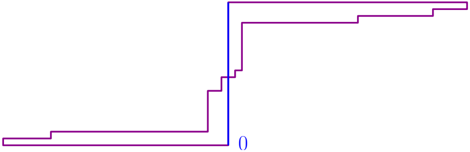

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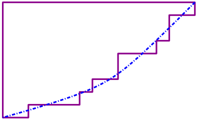
Our topic today



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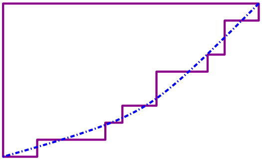
Our limit regime for today

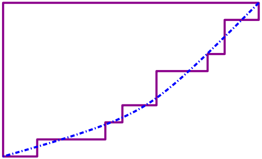


All rows grow linearly in N.

Rescaled profile approximates a **limit shape**.

- Preserves natural symmetry between rows and columns
- No degeneration to $S(n)$, random matrices or $U(\infty)$.
- Connections to statistical mechanics models: lozenge tilings, six–vertex model, percolation in a strip.

Law of Large Numbers for tensor products



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$$T_{\lambda(N)} \otimes T_{\lambda'(N)} = \bigoplus_{\mu} c_{\mu}^{\lambda(N),\lambda'(N)} T_{\mu}$$

How does signature of **typical irreducible component** look like?

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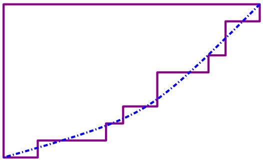

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Assumption. Suppose that for bounded piecewise continuous f, g

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\lambda_i(N) + N - i}{N} - f\left(\frac{i}{N}\right) \right| = 0, \qquad \sup_{i,N} \left| \frac{\lambda_i(N)}{N} \right| < \infty$$

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The convergence and the operation $(f, g) \mapsto h$ will be explained.

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$$T_{\lambda(N)}\Big|_{U(k)} = \bigoplus_{\mu} c_{\mu}^{\lambda(N)} T_{\mu} \quad \to \quad \mathrm{Prob}(\mu) = \frac{c_{\mu}^{\lambda(N)} \dim(\mu)}{\dim(\lambda(N))}.$$

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Theorem. (Gorin–Panova–2013, Bufetov–Gorin–2013) Then (scaled by N) **random** profile of μ converges to the **deterministic**

1. Constant $-\frac{1}{2} + \int_0^1 f(t)dt$ if k is fixed.
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Remark. For restrictions (but not for tensor products!) concentration of measure can be also deduced from the variational principle of Cohn–Kenyon–Propp and Kenyon–Okounkov–Sheffield for **random lozenge tilings**.

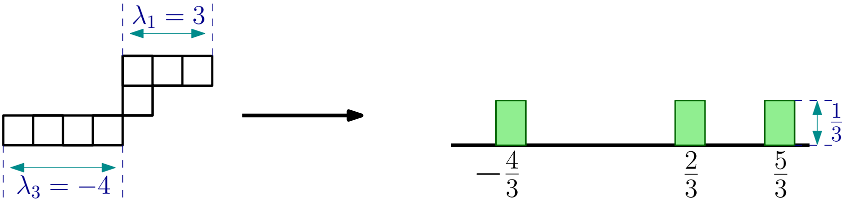
Law of Large Numbers: formulas

Our aim is to explain the operations $(f, g) \mapsto h$ and $f \mapsto f_\alpha$ on the limit profiles, arising from tensor products and restrictions, respectively.

Law of Large Numbers: formulas

Profile of a signature $\mapsto$ counting measure:

$\lambda \to$ (prob. measure on $\mathbb{R}$) $\qquad \mathbf{m}[\lambda] = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right)$



The convergence of $\lambda(N)$ to f implies **weak convergence** of $\mathbf{m}[\lambda(N)]$ to the limit measure $\mathbf{m}[f]$.

Remark. This definition *depends* on the group. We present the case of the unitary group $U(N)$ here.

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The convergence of $\lambda(N)$ to f implies **weak convergence** of $\mathbf{m}[\lambda(N)]$ to the limit measure $\mathbf{m}[f]$.

$$m_k(\mathbf{m}) = \int_{\mathbb{R}} x^k \mathbf{m}(dx).$$

$$S_{\mathbf{m}(u)} = z + m_1(\mathbf{m})z^2 + m_2(\mathbf{m})z^3 + \dots,$$

$$R_{\mathbf{m}}^{quant}(u) = \frac{1}{(S_{\mathbf{m}}(u))^{-1}} - \frac{1}{1 - e^{-u}},$$

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Remark. $R^{quant}_{\mathbf{m}}(u) = R_{\mathbf{m}}(u) + \frac{1}{u} - \frac{1}{1-e^{-u}}$, where $R_{\mathbf{m}}(u)$ is Voiculescu R–function (a **free probability** analogue of characteristic function) and $\frac{1}{1-e^{-u}} - \frac{1}{u}$ is R–function for the uniform measure on $[0, 1]$.

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Theorem. (Bufetov–Gorin–2013) Suppose that $\lambda(N)$, $\lambda'(N)$ converge to the limit profiles encoded by the measures $\mathbf{m}$, $\mathbf{m}'$. Then the **random** probability measure corresponding to μ converges to the **deterministic** measure $\mathbf{m} \otimes \mathbf{m}'$, such that

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$$N \to \infty,\ k = \alpha N.$$

Theorem. Suppose that $\lambda(N)$ converges to the limit profile encoded by the measure $\mathbf{m}$. Then the **random probability measure** corresponding to μ (scaled by αN) converges to the **deterministic measure** $\mathbf{m}_{\alpha}$, such that

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Law of Large Numbers for Restrictions: example.

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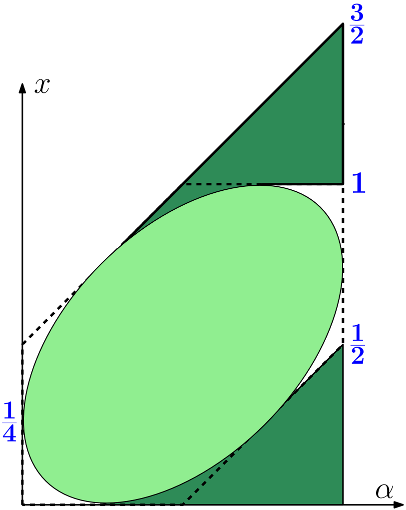

Plot of support of pushforward of $\alpha \mathbf{m}_\alpha$ under $x \mapsto \alpha x$, i.e. limit for the random measure

$$\mathbf{m}[\mu] = \frac{1}{N} \sum_{i=1}^{N\alpha} \delta\left(\frac{\mu_i + N\alpha - i}{N}\right).$$

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For $0 < \alpha < 1$, density of $\mathbf{m}_\alpha$ is

$$\begin{cases} 0, & x > 1 + \frac{1}{2\alpha}, \\ 1, & x < 1 + \frac{1}{2\alpha}, x > \frac{1}{\alpha}, \\ 1, & x > 0, x < 1 - \frac{1}{2\alpha}, \\ 0, & x < 0, \\ \frac{1}{\pi} \arccos(\phi), & \text{otherwise.} \end{cases}$$

$$\phi = \frac{3/4 - (1 - \alpha x)((\frac{1}{2} + \alpha) - \alpha x) - \alpha x(\alpha x + (\frac{1}{2} - \alpha))}{2\sqrt{\alpha x(1 - \alpha x)((\frac{1}{2} + \alpha)1 - \alpha x)(\alpha x + (\frac{1}{2} - \alpha))}}.$$

(ϕ is set to 0 or π if argument is out of $[-1, 1]$)

Tensor products and restrictions

Summary. (Bufetov–Gorin–2013) As $N \to \infty$ the **random** probability measures corresponding to tensor products and restrictions to smaller subgroups of irreducible representations of $U(N)$ converges to **deterministic** measures $\mathbf{m} \otimes \mathbf{m}'$, $\mathbf{m}_\alpha$, such that

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Remark 1. Tensor powers are related to restrictions through

$$\mathbf{m}^{\otimes k} = \mathbf{m}_{1/k}.$$

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Remark 2. The linearization function R^{quant} is not unique. One of its forms can be guessed as a limit

$$\lim_{N\to\infty} \frac{1}{N} \log \left(\frac{s_{\lambda(N)}(x, 1^{N-1})}{s_{\lambda(N)}(1^N)} \right)$$

computed in (Guionnet–Maida–2005), (Gorin–Panova–2013).

Tensor products and restrictions

Summary. (Bufetov–Gorin–2013) As $N \to \infty$ the **random** probability measures corresponding to tensor products and restrictions to smaller subgroups of irreducible representations of $U(N)$ converges to **deterministic** measures $\mathbf{m} \otimes \mathbf{m}'$, $\mathbf{m}_\alpha$, such that

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Remark 3. (Borodin–Bufetov–Olshanski–2013) In the context of the limit shape theorem for the restrictions of characters of the **infinite–dimensional** unitary group $U(\infty)$ the same operation on the measures corresponds to the unions of the Voiculescu parameters for extreme characters.

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Remark 4. We call the operation $(\mathbf{m}, \mathbf{m}') \mapsto \mathbf{m} \otimes \mathbf{m}'$ **quantized free convolution**. Why?

Free Convolution

Let $A[N]$ and $B[N]$ be independent *uniformly random* Hermitian matrices with **fixed** eigenvalues $\{a_i[N]\}$, $\{b_i[N]\}$.

Free Convolution

Let $A[N]$ and $B[N]$ be independent *uniformly random* Hermitian matrices with **fixed** eigenvalues $\{a_i[N]\}$, $\{b_i[N]\}$.

Theorem–Definition. (Following Voiculescu and others) Suppose

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta(a_i[N]) \to \mathbf{m}_A, \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta(b_i[N]) \to \mathbf{m}_B,$$

Let $C[N] = A[N] + B[N]$ and $D_\alpha[N] =$"top left αN corner of $A[N]$, then their spectral measures converge to deterministic **free convolution** $\mathbf{m}_A \boxplus \mathbf{m}_B$ and **free projection** $\mathbf{m}_{\alpha A}$. Moreover,

$$R_{\mathbf{m}_A \boxplus \mathbf{m}_B}(u) = R_{\mathbf{m}_A}(u) + R_{\mathbf{m}_B}(u), \quad R_{\mathbf{m}_{\alpha A}}(u) = \frac{1}{\alpha} R_{\mathbf{m}_A}(u).$$

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There is a way to **degenerate** quantized free convolution into (conventional) free convolution.

q.FC $\to$ Free Convolution

Quantized free convolution can be degenerated into free convolution.

1. **"Semiclassical limit"**: Large representations of a (fixed) Lie group behave as group-invariant measures on (dual to) the Lie algebra. (N is kept finite)

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2. Limit transition between quantized free convolution and free convolution by **measure scaling**:

$$\lim_{L\to+\infty} \frac{R^{quant}_{\mathbf{m}\cdot L}\left(\frac{z}{L}\right)}{L} = R_{\mathbf{m}}(z),$$

$$(\mathbf{m}\cdot L)(A) = \mathbf{m}\left(A/L\right), \quad A\subset\mathbb{R}, \quad L>0.$$

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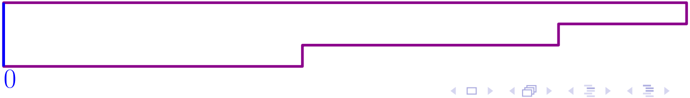
3. Law of Large Numbers for tensor products of representations with **superlinearly growing** ("very thick") signatures (Biane-1996, Collins–Sniady–2009).



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$$R^{quant}_{\mathbf{m}}(z) = R_{\mathbf{m}}(z) - R_{u[0,1]}(z)$$

$$R^{quant}_{\mathbf{m}^1 \otimes \mathbf{m}^2}(z) = R^{quant}_{\mathbf{m}^1}(z) + R^{quant}_{\mathbf{m}^2}(z)$$

Can we **guess** the appearance of $u[0,1]$?

The following heuristics is due to Biane.

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This is only heuristics.

Quantized Free Convolution and asymptotic freeness

The **direct** connection to free probability is restored if we slightly change a point of view.

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- Representation of the group $U(N)$ of unitary matrices..

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- **Reminder**: Lie algebra $\mathfrak{gl}(N)$ is spanned by matrix units E_{ij}, $1 \leq i, j \leq N$.

$$\text{E.g. for } N = 4 \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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- **Reminder**: Associative algebra $\mathcal{U}(\mathfrak{gl}(N))$ is spanned by *formal generators* E_{ij} subject to

$$E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_j^k E_{il} - \delta_i^l E_{kj}.$$

Quantized Free Convolution and asymptotic freeness

Definition. $\mathcal{E}(N)$ — $N \times N$ **matrix over** $\mathcal{U}(\mathfrak{gl}(N))$.

$$\mathcal{E}(N) = \begin{pmatrix} E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & \ddots & & E_{2N} \\ \vdots & & & \vdots \\ E_{N1} & E_{N2} & \dots & E_{NN} \end{pmatrix} \in \mathcal{U}(\mathfrak{gl}(N)) \otimes \mathrm{Mat}_{N\times N}$$

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Lemma. (Perelomov–Popov–68) **Center** of $\mathcal{U}(\mathfrak{gl}(N))$ is spanned by

$$X_p = \mathrm{Trace}\left(\mathcal{E}(N)^p\right) = \sum_{i_1,\dots,i_p=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_p i_1}$$

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Later $\mathcal{E}(N)$ played an important role in representation theory.
It appears in our problem as well.

Quantized Free Convolution and asymptotic freeness

Assumption. Suppose that for bounded piecewise continuous f, g

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left|\frac{\lambda_i(N)+N-i}{N}-f\left(\frac{i}{N}\right)\right|=0,\qquad \sup_{i,N}\left|\frac{\lambda_i(N)}{N}\right|<\infty$$

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Conjecture. Suppose $\mathcal{E}(\lambda;N)$ is $\mathcal{E}(N)$ acting in the first component of $T_{\lambda(N)}\otimes T_{\lambda'(N)}$ and similarly for $\mathcal{E}(\lambda';N)$. Then the elements $\frac{1}{N}\mathcal{E}(\lambda(N))$ and $\frac{1}{N}\mathcal{E}(\lambda'(N))$ are **asymptotically free.**

(As elements of $\mathrm{End}_{\mathbb{C}}(T_{\lambda(N)})\otimes\mathrm{End}_{\mathbb{C}}(T_{\lambda'(N)})\otimes\mathrm{Mat}_{N\times N}$.)

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Theorem. (Bufetov–Gorin–2013) Asymptotically the measure describing the spectrum of the sum $\frac{1}{N}\mathcal{E}(\lambda(N)) + \frac{1}{N}\mathcal{E}(\lambda'(N))$ is given by the **free convolution** of those of $\frac{1}{N}\mathcal{E}(\lambda(N))$ and $\frac{1}{N}\mathcal{E}(\lambda'(N))$.

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Remark. Related results appeared

- (Biane–1998) For representations of symmetric groups.
- (Biane–1996, Collins–Sniady-2009) For reps of $U(N)$ with "very thick" signatures, which are well-approximated by the random matrix objects.

Quantized Free Convolution and asymptotic freeness

Theorem. (Perelomov–Popov–1968) Description of the spectrum

$$\mathrm{Trace}_{\mathbb{C}^N}\mathrm{Trace}_{T_\lambda}\left(\frac{1}{N}E(\lambda(N))\right)^k = \int_{\mathbb{R}} x^k d\mathbf{m}_{PP}[\lambda],$$

$$\mathbf{m}_{PP}[\lambda] = \frac{1}{N}\sum_{i=1}^{N}\left(\prod_{j\neq i}\frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)}\right)\delta\left(\frac{\lambda_i + N - i}{N}\right).$$

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$$Q(\rho_1) \boxplus Q(\rho_2) = Q(\rho_1 \otimes \rho_2).$$

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In addition to being highly non-linear/non-trivial, $\rho \mapsto Q(\rho)$ is injective, but **not surjective**. Thus q.FC and FC are **not** reduced one to another.

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Q is a relative of the map between **Markov moment problem** and **Hausdorff moment problem**.

Further directions.

1. Characters of the **infinite–dimensional unitary group** $U(\infty)$ and (quantized free convolution–) infinitely–divisible measures.
2. Perelomov–Popov operators and measures $\longleftrightarrow$ results of Biane on operations on reps of $S(n)$ and Kerov transition measure.
3. Results for $SO(2N+1)$, $Sp(2N)$, $SO(2N)$: one needs to **double** the signatures by reflection. (Mysterious identities.)
4. Restrictions of representations can be linked to random **lozenge tilings** of planar domains.
5. Our methods: asymptotics of characters via *integral representations* and steepest descent; use of *differential operators* diagonalizable by characters.

Our method

$$T_{\lambda(N)} \otimes T_{\lambda'(N)} = \bigoplus_{\mu} c_{\mu}^{\lambda(N),\lambda'(N)} T_{\mu} \quad \rightarrow$$

$$\mathrm{Prob}(\mu) = \frac{c_{\mu}^{\lambda(N),\lambda'(N)} \dim(\mu)}{\dim(\lambda(N)) \dim(\lambda'(N))}.$$

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Can be encoded via

$$\frac{s_{\lambda(N)}(x_1, \ldots, x_N) s_{\lambda'(N)}(x_1, \ldots, x_N)}{s_{\lambda(N)}(1^N) s_{\lambda'(N)}(1^N)} = \sum_{\mu} \text{Prob}(\mu) \frac{s_{\mu}(x_1, \ldots, x_N)}{s_{\mu}(1^N)}$$

Our method

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$$\frac{s_{\lambda(N)}(x_1,\dots,x_N)s_{\lambda'(N)}(x_1,\dots,x_N)}{s_{\lambda(N)}(1^N)s_{\lambda'(N)}(1^N)} = \sum_{\mu} \mathrm{Prob}(\mu)\frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1^N)}$$

Apply the **differential operator** $(D_k)^m$ and evaluate at $x_i = 1$.

$$D_k = \prod_{i<j}\frac{1}{x_i - x_j}\sum_{i=1}^{N}\left(x_i\frac{\partial}{\partial x_i}\right)^k \prod_{i<j}(x_i - x_j),$$

using

$$s_\mu(x_1,\dots,x_N) = \frac{\det\left(x_i^{\mu_j+N-j}\right)_{i,j=1}^{N}}{\prod_{i<j}(x_i - x_j)}.$$

(These are radial parts of certain elements of the center of $\mathcal{U}(\mathfrak{gl}(N))$.)

Our method

$$D_k = \prod_{i<j} \frac{1}{x_i - x_j} \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \prod_{i<j} (x_i - x_j),$$

$$\mathrm{Prob}(\mu) = \frac{c_{\mu}^{\lambda(N),\lambda'(N)} \dim(\mu)}{\dim(\lambda(N)) \dim(\lambda'(N))}.$$

We obtain:

$$(D_k)^m \frac{s_{\lambda(N)}(x_1,\ldots,x_N) s_{\lambda'(N)}(x_1,\ldots,x_N)}{s_{\lambda(N)}(1^N) s_{\lambda'(N)}(1^N)} \Bigg|_{x_i=1} = \sum_{\mu} \left(\sum_{i=1}^{N} (\mu_i + N - i)^k \right)^m \mathrm{Prob}(\mu).$$

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Remark. An approach to study of measures through difference operators was used in (Borodin–Corwin–2011+), (Borodin–Corwin–Gorin–Shakirov–2013) in the framework of **Macdonald processes**.

Our method

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Now the right side divided by N^{km} approximates **expectations of the moments** of (a priori random) limit measure.

$$\mathbb{E}\left(\int x^k \mathbf{m}(dx)\right)^m$$

Our method

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$$= \sum_{\mu}\left(\sum_{i=1}^{N}(\mu_i+N-i)^k\right)^m \mathrm{Prob}(\mu).$$

Asymptotic analysis of normalized polynomials in the left side is based on **integral representation** (Gorin–Panova–2013).

$$\frac{s_\lambda(x,1^{N-1})}{s_\lambda(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint_C \frac{x^z}{\prod_{i=1}^{N}(z-(\lambda_i+N-i))}dz,$$

And classical steepest descent method for the analysis of

$$\oint F(z)\exp(NG(z))dz, \quad N\to\infty$$

Our method

$$(D_k)^m \frac{s_{\lambda(N)}(x_1,\dots,x_N) s_{\lambda'(N)}(x_1,\dots,x_N)}{s_{\lambda(N)}(1^N) s_{\lambda'(N)}(1^N)}\Bigg|_{x_i=1} = \sum_{\mu} \left(\sum_{i=1}^{N} (\mu_i + N - i)^k \right)^m \mathrm{Prob}(\mu).$$

Asymptotic analysis of normalized polynomials in the left side is based on **determinantal formulas** of (Gorin–Panova–2013).

$$\frac{s_\lambda(x_1,\dots,x_k,1^{N-k})}{s_\lambda(1^N)} = \frac{1}{\prod_{i<j}(x_i - x_j)} \prod_{i=1}^{k} \frac{(N-i)!}{(x_i - 1)^{N-k}}$$

$$\times \det \left[\left(x_i \frac{\partial}{\partial x_i} \right)^{k-j} \right]_{i,j=1}^{k} \left(\prod_{j=1}^{k} \frac{s_\lambda(x_j, 1^{N-1})}{s_\lambda(1^N)} \frac{(x_j - 1)^{N-1}}{(N-1)!} \right),$$

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Alexey Bufetov, V.G., *Representations of classical Lie groups and quantized free convolution,* arXiv:1311.5780