Bi-free Convolution in the Plane and the Simplest Bi-free R-transform

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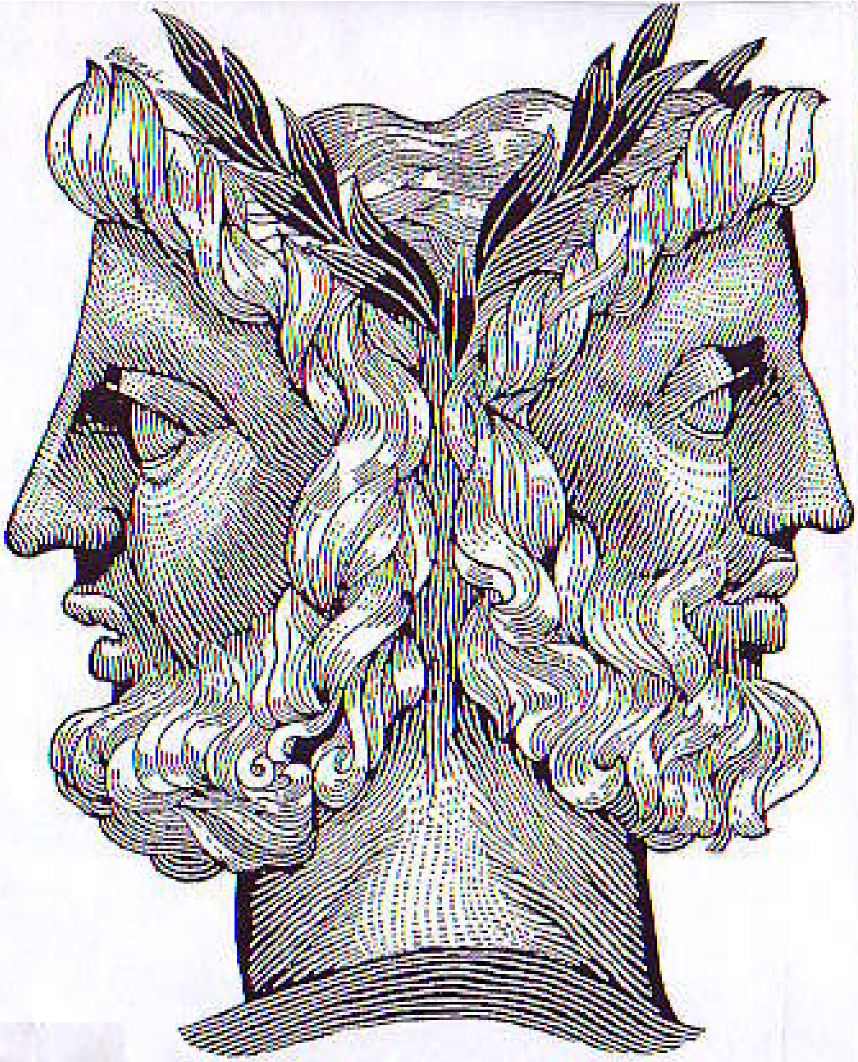
Free Probability for Pairs of Faces

(Free Probability for Systems with Left and Right Variables)



Janus

2 faces

Past and Future

Transition

[Left Var, Right Var] = 0

Bipartite System

Free Product of Vector Spaces with specified State Vectors

$$\mathcal{X}_\iota = \overset{\circ}{\mathcal{X}}_\iota \oplus \mathbb{C}\xi_\iota$$

$$\mathcal{X} = \mathbb{C}\xi \oplus \underbrace{\bigoplus_{n \geq 1} \bigotimes_{\iota_1 \neq \cdots \neq \iota_n} \overset{\circ}{\mathcal{X}}_{\iota_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{X}}_{\iota_n}}_{\overset{\circ}{\mathcal{X}}}$$

$$(\mathcal{X}, \overset{\circ}{\mathcal{X}}, \xi) = \mathop{\ast}_{\iota \in I} (\mathcal{X}_\iota, \overset{\circ}{\mathcal{X}}_\iota, \xi_\iota)$$

$\varphi_\xi : \mathcal{L}(\mathcal{X}) \to \mathbb{C}$, $T\xi \in \varphi_\xi(T)\xi \oplus \overset{\circ}{\mathcal{X}}$.

Left and Right Factorizations

$$V_\iota : \mathcal{H}_\iota \otimes \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{\iota \neq \iota_1 \neq \cdots \neq \iota_n} \mathring{\mathcal{H}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{\iota_n}\Big) \to \mathcal{H}$$

$$W_\iota : \Big(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigotimes_{\iota_1 \neq \cdots \neq \iota_n \neq \iota} \mathring{\mathcal{H}}_{\iota_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{\iota_n}\Big) \otimes \mathcal{H}_\iota \to \mathcal{H}$$

$$T \in \mathcal{L}(\mathcal{H}_\iota)$$

$$\lambda_\iota(T) = V_\iota (T \otimes I) V_\iota^{-1} \in \mathcal{L}(\mathcal{H})$$

$$\rho_\iota(T) = W_\iota (I \otimes T) W_\iota^{-1} \in \mathcal{L}(\mathcal{H})$$

$$[\lambda_i(T), \rho_j(S)] = \delta_{ij} \cdot [T,S] \oplus 0 \,.$$

$(\mathcal{A}, \varphi)$ noncommutative probability space

Pair of Faces in $(\mathcal{A}, \varphi)$

$(\mathcal{B}, \beta)$ left face , right face $(\mathcal{C}, \gamma)$

$$\begin{tikzcd} & \mathcal{A} & \\ \mathcal{B} \arrow[ur, "\beta"] & & \mathcal{C} \arrow[ul, "\gamma"'] \end{tikzcd}$$

β, γ unital homomorphisms

$\mathcal{B}, \mathcal{C}$ unital algebras

Included faces $\mathcal{B} \subset \mathcal{A} \supset \mathcal{C}$.
(β, γ are the inclusions)

2-faced family of noncommutative random variables in $(\mathcal{A}, \varphi)$
$$((b_\iota)_{\iota \in I}, (c_j)_{j \in J}) \text{ in } \mathcal{A}$$
[Corresponds to
$$\beta : \mathbb{C}\langle X_\iota \mid \iota \in I \rangle \longrightarrow \mathcal{A}, \quad \beta(X_\iota) = b_\iota$$
$$\gamma : \mathbb{C}\langle Y_j \mid j \in J \rangle \longrightarrow \mathcal{A}, \quad \gamma(Y_j) = c_j]$$

Bi-freeness of a family of pairs of faces

$((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ in (A, φ):

$\exists\ (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota), \iota \in I$, $(\mathcal{X}, \mathring{\mathcal{X}}, \xi) = \underset{\iota \in I}{\ast} (\mathcal{X}_\iota, \mathring{\mathcal{X}}_\iota, \xi_\iota)$

$l_\iota : B_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$, $r_\iota : C_\iota \longrightarrow \mathcal{L}(\mathcal{X}_\iota)$

unital homomorphisms, so that

$$\varphi \circ \pi = \varphi_\xi \circ \tilde{\pi}$$

$\pi : \underset{\iota \in I}{\ast} (B_\iota \ast C_\iota) \longrightarrow A$, $\pi|B_\iota = \beta_\iota$, $\pi|C_\iota = \gamma_\iota$

$\tilde{\pi} : \underset{\iota \in I}{\ast} (B_\iota \ast C_\iota) \longrightarrow \mathcal{L}(\mathcal{X})$, $\tilde{\pi}|B_\iota = \lambda_\iota \circ l_\iota$, $\tilde{\pi}|C_\iota = \rho_\iota \circ r_\iota$.

<u>Remarks</u>: 1° If $((B_\iota,\beta_\iota),(C_\iota,\gamma_\iota))_{\iota\in I}$ bi-free in $(\mathcal{A},\varphi)$, joint distribution $\varphi\circ\tilde{\pi}$ obtained also as $\varphi_{\xi'}\circ\tilde{\pi}'$ for any other $(\mathcal{X}'_\iota,\mathring{\mathcal{X}}'_\iota,\xi'_\iota)$, ℓ'_ι, r'_ι so that

$$\varphi\circ\pi_\iota = \varphi_{\xi'_\iota}\circ\pi'_\iota$$

$\pi_\iota : B_\iota * C_\iota \to \mathcal{A}$, $\pi_\iota|_{B_\iota} = \beta_\iota$, $\pi_\iota|_{C_\iota} = \gamma_\iota$

$\pi'_\iota : B_\iota * C_\iota \to \mathcal{L}(\mathcal{X}'_\iota)$, $\pi'_\iota|_{B_\iota} = \ell'_\iota$, $\pi'_\iota|_{C_\iota} = r'_\iota$

2°. $((B_\iota, \beta_\iota), (C_\iota, \gamma_\iota))_{\iota \in I}$ bi-free in $(\mathcal{A}, \varphi)$

then $(\beta_\iota(B_\iota))_{\iota \in I}$ free in $(\mathcal{A}, \varphi)$

$\iota \neq j \Rightarrow \beta_\iota(B_\iota), \gamma_j(C_j)$ classically independent in $(\mathcal{A}, \varphi)$.

3°. bi-freeness has the necessary properties to be used as an independence relation in a noncommutative probability theory with left and right variables i.e. two-faced families.

4°. C^*– bi-freeness, W^*– bi-freeness
bi-free products of states etc.
bi-free convolution operations
(additive, multiplicative)

$\mu \boxplus\boxplus \nu$, $\mu \boxtimes\boxtimes \nu$

Bi-freeness Examples

I. Groups $(G_\iota)_{\iota\in I}$, $G = \ast_{\iota\in I} G_\iota$.

$L_\iota : \mathbb{C}[G_\iota] \to \mathcal{L}(\mathbb{C}[G])$

$R_\iota : \mathbb{C}[G_\iota]^{op} \to \mathcal{L}(\mathbb{C}[G])$

restrictions of left and right regular representations

$((\mathbb{C}[G_\iota], L_\iota), (\mathbb{C}[G_\iota]^{op}, R_\iota))_{\iota\in I}$

bi-free family of faces in $(\mathbb{C}[G], \tau)$.

v. Neumann trace

II. Left and right creation and annihilation operators on the full Fock space.

$\mathcal{H}$ complex Hilbert sp. $(e_\iota)_{\iota\in I}$ ONB

$\mathcal{T}(\mathcal{H}) = \mathbb{C}1 \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$

$l_\iota \xi = e_\iota \otimes \xi$, $r_\iota \xi = \xi \otimes e_\iota$, $\xi \in \mathcal{T}(\mathcal{H})$.

$\omega(T) = \langle T1, 1\rangle$ on $\mathcal{B}(\mathcal{T}(\mathcal{H}))$

$((l_\iota, l_\iota^*), (r_\iota, r_\iota^*))_{\iota\in I}$

bi-free in $(\mathcal{B}(\mathcal{T}(\mathcal{H})), \omega)$.

Bi-free Cumulants

$z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ 2-faced family of r.v. in (A, φ)

Moments $\varphi(z_{\alpha(1)} \cdots z_{\alpha(n)})$, $\alpha : \{1, \ldots, n\} \to I \amalg J$

R_α polynomial in commuting variables $X_{\alpha(k_1) \cdots \alpha(k_r)}$, $1 \le k_1 < \cdots < k_r \le n$.

homogeneous $\deg = n$, $\deg X_{\alpha(k_1) \cdots \alpha(k_r)} = r$.

$$R_\alpha(z) = R_\alpha(\varphi(z_{\alpha(k_1)} \cdots z_{\alpha(k_n)}) \mid 1 \le k_1 < \dots < k_n \le n\}$$

R_α bi-free cumulant, exists & unique so that:

1°. coefficient of $X_{\alpha(1)\dots\alpha(n)} = 1$

2°. z', z'' bi-free in (A, φ), then

$$R_\alpha(z') + R_\alpha(z'') = R_\alpha(z' + z'').$$

$$\alpha\Pi_m = \{(\alpha(k_1), \dots \alpha(k_r)) \mid 1 \le k_1 < \dots < k_r \le m,\ 1 \le r \le n\}$$

$$\mu_{z,\alpha} = \left(\varphi(z_{\alpha(1)} \cdots z_{\alpha(k_r)})\right)_{(\alpha(k_1), \dots \alpha(k_r)) \in \alpha\Pi_m}$$

$$(\mu_{z',\alpha}, \mu_{z'',\alpha}) \longrightarrow \mu_{z'+z'',\alpha}$$

polynomial abelian group law on $\mathbb{C}^{\alpha\Pi_m}$

$$\mathbb{C}^{\alpha\Pi_m} \xrightarrow{\exp} \mathbb{C}^{\alpha\Pi_m} \quad \text{isomorphism}$$

(Lie algebra, $+$) $\qquad$ $\boxplus\ \boxplus_n$ law

$\log = (\exp)^{-1}$ yields cumulants

One Variable free R-transform

$a \in (A, \varphi)$

$R_n(a)$ n-th free cumulant

leading term $\varphi(a^n)$

$$\mathcal{R}_a(z) = \sum_{n \geq 1} z^{n-1} R_n(a)$$

$\mathcal{R}$-transform

Fact (1986)

$$G_a(z) = \sum_{n \geq 0} z^{-n-1} \varphi(a^n) = \varphi((z1 - a)^{-1})$$

$$K_a(z) = z^{-1} + R_a(z)$$

then $G_a(K_a(z)) = z$ (near 0)

or equivalently

$K_a(G_a(z)) = z$ (near ∞)

(formal or germs of holo)

Two Variables bi-free partial R-transform

$a, b \in (\mathcal{A}, \varphi)$

$R_{m,n}(a,b)$ bi-free cumulant

leading term $\varphi(a^m b^n)$

$$R_{a,b}(z,w) = \sum_{\substack{m \geq 0, n \geq 0 \\ m+n \geq 1}} R_{m,n}(a,b) z^m w^n$$

partial bi-free R-transform
(shift by +1 in exponents)

<u>Fact</u> (2013)

$$G_{a,b}(z,w) = \sum_{m \geq 0, n \geq 0} z^{-m-1} w^{-n-1} \varphi(a^m b^n)$$

$$= \varphi\left((z1-a)^{-1}(w1-b)^{-1}\right)$$

then

$$R_{a,b}(z,w) = 1 + z R_a(z) + w R_b(w)$$

$$- \frac{z\,w}{G_{a,b}(K_a(z), K_b(w))}$$

(formal or germs of holo near $(0,0)$)

Proof of 2013 Fact uses Haagerup alternative Proof of 1986 Fact.

Distributions for which $\boxplus\boxplus$ can be computed adding the $R_{a,b}(z,w)$:

distributions completely determined by two-bands moments $\varphi(a^m b^n)$ and which after $\boxplus\boxplus$ still similarly determined by two-bands moments

Bi-partite systems $((a_i)_{i \in I}, (b_j)_{j \in J})$

$$[a_i, b_j] = 0, \; i \in I, \; j \in J$$

Distribution determined by

$\varphi(\underset{\substack{\text{left}\\\text{monomial}}}{L} \; \underset{\substack{\text{right}\\\text{monomial}}}{R})$ two band moments (starting left)

Bi-free addition of bi-partite systems gives bi-partite system

$a = a^*, \ b = b^*, \ [a, b] = 0$

joint distribution given by probability measure μ on $\mathbb{R}^2$ (compact support)

$$G_{a,b}(z, w) = G_\mu(z, w) = \iint \frac{d\mu(s, t)}{(s-z)(t-w)}$$

$\mu_k = pr_k * \mu$

$$G_a(z) = G_{\mu_1}(z) = \int \frac{d\mu_1(s)}{z - s}$$

$$G_b(w) = G_{\mu_2}(w) = \int \frac{d\mu_2(t)}{w - t}$$

$$G_{\mu_1}(K_{\mu_1}(z)) = z, \quad G_{\mu_2}(K_{\mu_2}(w)) = w$$

$$R_\mu(z,w) = 1 + zR_{\mu_1}(z) + wR_{\mu_2}(w) - \frac{zw}{G_\mu(K_{\mu_1}(z), K_{\mu_2}(w))}$$

Bi-free convolution $\mu \boxplus\boxplus \nu$

$$R_{(\mu \boxplus\boxplus \nu)_k}(z) = R_{\mu_k}(z) + R_{\nu_k}(z) \quad k=1,2$$

$$R_{\mu \boxplus\boxplus \nu}(z,w) = R_\mu(z,w) + R_\nu(z,w)$$

get $G_{\mu \boxplus\boxplus \nu}(z,w)$ and then $\mu \boxplus\boxplus \nu$.

Systems with Rank ≤ 1 Commutation

$(\mathcal{A}, P, \varphi)$ implemented noncommutative probability space

$P = P^2$, $P\mathcal{A}P = \mathbb{C}P$ minimal projection

$\varphi(a)P = PaP \quad a \in \mathcal{A}$

$((a_i)_{i \in I}, (b_j)_{j \in J})$

$[a_i, b_j] = \lambda_{ij} P$

Distribution determined by 2-bands moments and $(\lambda_{ij})_{i \in I, j \in J}$

<u>Example</u> Bi-free Gaussian systems

$\mathcal{H}$, $\mathcal{T}(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$, $\mathcal{H}^{\otimes 0} = \mathbb{C}1$

$P = \langle \cdot 1, 1 \rangle 1$, $\mathcal{A} = \mathcal{B}(\mathcal{T}(\mathcal{H}))$, $\varphi = \langle \cdot 1, 1 \rangle$

$h, h^* : I \amalg J \longrightarrow \mathcal{H}$

$((a_i)_{i \in I}, (b_j)_{j \in J})$

$$a_i = \ell(h(i)) + \ell^*(h^*(i)), \; i \in I$$
$$b_j = r(h(j)) + r^*(h^*(j)), \; j \in J$$

$$[a_i, b_j] = (\langle h(j), h^*(i) \rangle - \langle h(i), h^*(j) \rangle) P.$$