Free Transport for convex potentials.

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Berkeley, March 2014

Overview

1. Introduction to classical and free transport
   - Classical transport
   - Previous results on free monotone transport
   - First formal equation for free infinitesimally monotone transport
2. Classes of non-commutative functions.
   - Analytic functions with expectation.
   - Haagerup tensor product valued free difference quotient
   - C^k-functions with expectation and stability properties.
3. Regularity of diffusion and transport
   - Notions of non-commutative convexity and uniqueness of τ_V
   - Regularity of free SDEs
   - Construction of transport maps.

1.1 Classical Transport

- A transport map $F : \mathbb{R}^n \to \mathbb{R}^n$ between μ and ν is a map such that $F_* \mu = \nu$ i.e. for any h positive measurable
$$\int h(x) d\nu(x) = \int h(F(x)) d\mu(x).$$
- For $d\mu = \frac{1}{Z_V} \exp(-V(x))dx$ to $d\nu = \frac{1}{Z_W} \exp(-W(x))dx$. Let JF stand for the Jacobian (derivative) of F. Then the transport equation reads:
$$\det(JF(x)) = C \exp(W(F(x)) - V(x)).$$
- F is not determined by this equation (compose with measure preserving maps). If one looks for $F = \nabla g$ then one gets a more restrictive equation called Monge-Ampere equation :
$$\det(J\nabla g(x)) = C \exp(W(\nabla g(x)) - V(x)).$$
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1.1 Classical transport: infinitesimally monotone variant

- Take μ_t a path of measure, $\mu_0 = \mu, \mu_1 = \nu$, say $d\mu_t = \exp(-W_t(x))dx, W_t(x) = ((1-t)V(x)) + tW(x)$
One can look for transport maps F_t between μ_0 and μ_t.
Differentiating the transport equation, one gets :
$$Tr[J\dot{F}_t(x)(JF_t)^{-1}(x)] = (W-V)(F_t(x)) + \sum_i \partial_{x_i}(W_t)(F_t(x))\dot{F}_t^i(x).$$
- One can look for
$$\dot{F}_t = \nabla g_t(F_t(x)),$$
(infinitesimal monotonicity) so that $J\dot{F}_t = (J\nabla g_t(F_t(x)))J(F_t(x))$ and
$$(\Delta g_t)(F_t(x)) - \nabla W_t(F_t(x)).\nabla g_t(F_t(x)) = (W-V)(F_t(x))$$
which is linear involving the generator $L_{W_t} = \Delta - \nabla W_t.\nabla$ of the diffusion with stationary measure μ_t.
- This defines an evolution equation for F_t with g_t determined by a Laplace type PDE. We will try to solve the free analogue of this problem.

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1.2 Free Gibbs state with potentials

- Recall the free analogue of $d\mu = \frac{1}{Z}\exp(-V(x))dx$ is a law τ_V limit of $\frac{1}{Z_{N,V}}\exp(-NTr(V(M)))dLeb(M)$ on hermitian matrices.
- When V is a small perturbation of quadratic [Guionnet,Maurel-Segala] or (c,M) convex [Guionnet,Shlyakhtenko], τ_V is the unique solution of the Schwinger-Dyson equation

$$\tau \otimes \tau(\partial_i P) = \tau((\mathcal{D}_i V)P) \ \forall P \in \mathbb{C}\langle X_1, ..., X_n\rangle.$$

- $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_m)$ denotes the cyclic gradient which is linear and given, for any monomial P, by

$$\mathcal{D}_i P = \sum_{P = P_1 X_i P_2} P_2 P_1,$$

and where ∂_i denotes the free difference quotient ∂ such that :

$$\partial_i P = \sum_{P = P_1 X_i P_2} P_1 \otimes P_2 .$$

- Shlyakhtenko and Guionnet solve the following free Monge Ampere equation :

$$(1\otimes\tau+\tau\otimes 1)\, Tr \log \mathscr{J}\mathcal{D}g = \mathscr{S}\left[\{W(\mathcal{D}g(X))\} - \frac{1}{2}\sum X_j^2\right] \quad (1)$$

for transport between law with potential $V_0 = \frac{1}{2}\sum X_j^2$ and potential W ($\mathscr{S}$ means modulo commutators.)

Theorem (Guionnet-Shlyakhtenko 2012)

If $||W - V_0||_R$ small enough, there exists an analytic solution g to 1 such that $\mathcal{D}g(X_1, ..., X_n)$ is invertible analytic and have law τ_W if $(X_1, ..., X_n)$ are semicircular variables. Thus $W^*(\tau_{V_0}) \simeq W^*(\tau_W), C^*(\tau_{V_0}) \simeq C^*(\tau_W)$

- There are generalizations to the type III case [Brent Nelson]
- Goal : going beyond small perturbations of semicircular systems and get the isomorphism for W "regular convex."

1.2 Previous Results on free monotone transport

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If we look for a path of transport maps F_t from τ_V to $\tau_{W_t}, W_t = tW + (1-t)V$ with

$$\dot{F}_t = \mathcal{D}g_t(F_t),$$

to get an autonomous (infinitesimally monotone equation) one are reduced to take g_t satisfying :

$$(1 \otimes \tau + \tau \otimes 1)Tr(\mathscr{J}\mathcal{D}g_t(F_t)) \\ = \mathscr{S}\left[\sum_i \partial_i W_t(F_t(X)) \# \mathcal{D}_i g_t(X) + (W-V)(F_t)\right]$$

- Problem: the generator of the free diffusion is

$$L_{W_t} g = \sum_i m \circ (1 \otimes \tau \otimes 1)\partial_i \otimes 1 \partial_i g - \partial_i(g) \# \mathcal{D}_i W_t.$$

(with $(a \otimes b)\#c = acb$) and the equation above reads :

$$(L_{W_t} g)(F_t) + \sum_i (\tau \circ m \otimes 1) \circ ((123).\partial_i \otimes 1 \partial_i g)(F_t) = (W - V)(F_t) + [P, Q]$$

- We have to find a better adapted differential calculus to remove the supplementary second order term.

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   - C^k-functions (with expectation) and stability properties.
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2.1 Analytic functions with expectation

- If one looks at the construction of [Guionnet-Shlyakhtenko 2012], the transport map $\mathcal{D}g$ is of the form $\sum_{P_0,...,P_n} a_{P_0,...,P_n} P_0 \tau(P_1)...\tau(P_n)$, i.e. a Power series variant of the space considered in [Cebron2013] (when $B = \mathbb{C}$)

$$B\{X_1,...,X_n\} = B\langle X_1,...,X_n\rangle \otimes S(B\langle X_1,...,X_n\rangle)$$

- As in [Cebron2013], on analytic variants of $B\{X_1,...,X_n\}$, free diffusion equations are really semigroups with generator $\Delta_V = L_V + \delta_V$ with δ_V the derivation with

$$\delta_V(P_0) = 0, \delta_V(\tau(P_i)) = \tau(L_V(P_i)).$$

- If one considers also a full cyclic gradient

$$\mathscr{D}_i(P_0\tau(P_1)...\tau(P_n)) := \sum_{j=0}^{n} \mathcal{D}_i(P_j) \prod_{k\neq j} \tau(P_k),$$

then $\mathscr{D}_i\Delta_0 = \Delta_0\mathscr{D}_i$ and, formally, with $\dot{F}_t = \mathscr{D}g_t(F_t)$, then $(\Delta_{W_t} g)(F_t) = (W - V)(F_t) + \text{commutators}$

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2.1 Calculus on analytic functions with expectation

- Beyond the full cyclic gradients, there are 2 other natural "derivations" on $B\{X_1, ..., X_n\}$, the ordinary difference quotient

$$\partial_i(P_0\tau(P_1)...\tau(P_n)) = \partial_i(P_0)\tau(P_1)...\tau(P_n)).$$

  It is unavoidable since it is involved in the transport equation.
- There is also the full differential d_X as a function of $X_i's$ and a partial one d with a term involving ∂ removed.
- The second one a priori well commutes with conditional expectation on part of the variables, but for ∂_i, this depends on the space of value. This is okay on subspaces of $L^2(M) \otimes L^2(M)$ for free variables. But one may need a space where $(a \otimes b)\#c \mapsto acb$ can be extended to control Lipschitzness properties since

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2.2 Reminder on Haagerup tensor product of C^* algebras $A, B, C \dots$

.

$$||U||_{A\otimes_h B} = \inf\{||\sum_i x_i x_i^*||^{1/2}||\sum_i y_i^* y_i||^{1/2} \ : \ U = \sum_i x_i \otimes y_i\}.$$

Theorem (cf. e.g. Pisier's Book)

1. *For any C^* algebra C the multiplication map extends to a completely contractive map $m : C \otimes_h C \to C$*
2. *$\otimes_h$ is functorial and injective, i.e. for any C^* algebras $C \subset C', B$, we have $C \otimes_h B \subset C' \otimes_h B, B \otimes_h C \subset B \otimes_h C'$ isometrically. Moreover [Blecher] if M finite W^* alg $M \otimes_h M \subset M \otimes_{min} M \subset L^2(M \otimes M)$.*

One can also consider a cyclic variant

$$||\sum a_1 \otimes \dots \otimes a_n||_{A^{\otimes_{hc} n}} := \max_{\sigma \in \mathcal{C}_n} ||\sum a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)}||_{A^{\otimes_h n}}.$$

2.3 C^k-functions (with expectation)

- Fix $A = B * W^*(S_t, t > 0)$, $U \subset A_R^n = \{(X_1, ..., X_n), X_i = X_i^* \in A, \|X_i\| \leq R\}$. For $X \in U$ $P \in B\{X_1, ..., X_n\}$, one considers $P(X) = P(\tau_X, X)$.
- We define, for $l \leq k$, $C_c^k(A, U : B)$ as a completion of $B\langle X_1, ..., X_n\rangle$ for the norm (if U large enough) $\sup_{X \in U} \|P\|_{k,X}$with :

$$\|P\|_{k,X} = \left(\|P(X)\|_A + \sum_{l=1}^{k} \sum_{i \in [1,n]^l} \|\partial_i^l(P)(X)\|_{A^{\otimes_{hc}(l+1)}} \right).$$

- We define $C_{tr,c}^{k,l}(A, U : B)$ as the separation completion of the space of maps $X \in U \mapsto P(\tau_X) \in B\langle X_1, ..., X_n\rangle$, for $P \in B\{X_1, ..., X_n\}$ for the seminorm :

$$\|P\|_{k,l,U} = \sup_{X \in U} \|P\|_{k,X} + \sum_{p=1}^{l} \sup_{X \in U} \left(\sup_{H \in A_1^n} \|D_H^p P\|_{k-p,X} \right).$$

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2.3 C^k-functions and stability properties

- We define similarly a more ad hoc space $C^{k,l}_{tr,V,c}(A, U : B)$ as above for the seminorms :

$$||P||_{C^{k,l}_{tr,V,c}(A,U:B)} = ||\iota(P)||_{k,l,U} + ||\delta_V(P)||_{C^*_{tr}(A,U)}$$
$$+ \sum_{p=0}^{l-1} \sup_{\substack{Q \in (C^{k-1,p}_{tr}(A, U^{m-1} : B))_1 \\ m \geq 2}} ||\mathscr{D}_{i,Q(X')}(P)||_{k-1,p,U^m,tr}.$$

- We have a stability by composition :

Lemma

Wih conditions on $U \subset A^n_R, U' \subset A^n_S$ $(P, Q_1, ..., Q_n) \mapsto P(Q_1, ..., Q_n)$ extends continuously to $Q_1, ..., Q_n \in C^{k,l}_{tr}(A, U : B)$ with $||Q_i||_{0,0,U} < S$ and any $P \in C^{k,l}_{tr,c}(A, U' : B)$ with value in $C^{k,l}_{tr,c}(A, U : B)$, Lipschitz in Q if $P \in C^{k+1,l+1}_{tr,c}$. and also extends to $C^k_c(A, U' : B) \times (C^{k,l}_{tr,V}(A, U : B, E_D))^n \to C^{k,l}_{tr,V}$ for any $l \geq 1$.

Let $\mathcal{B} = B * W^*(S_t, t > 0)$ and for $\mathscr{S} = \{S_t, t > 0\}$, let $C^{k,l}_{tr,V,c}(A, U : B, \mathscr{S})$ the closure of elements coming from $B\{X_1, ..., X_n, S_t, t > 0\}$ in $C^{k,l}_{tr,V,c}(A, U : \mathcal{B})$ then we have a stability by conditional expectation.

Proposition

For any k, l and $U \subset A^n_{R,conj}$ (resp. $U \subset A^n_{R,conj2}$, if $k \geq 4$) $E_B : \mathscr{B} \to B$ gives a contraction $C^{k,l}_{tr,V}(A, U : \mathscr{B}, \mathscr{S}) \to C^{k,l}_{tr,V}(A, U : B)$. and we have compositions on $C^{k,l}_{tr,V}(A, U : \mathscr{B} : \mathscr{S}))$ as before.

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- For $V \in \mathbb{C}\langle X_1, ..., X_n\rangle$, convexity of $Tr(V)$ on all matrix spaces is in general not enough. In [GuionnetShlyakhtenkoGAFA], is considered the notion of (c,M) covexity (which is not clearly stable by cyclic perturbation), we will prefer a variant based on Haagerup tensor product.

Definition

$V = V^* \in C_c^2(A, U : B)$, is said generalized (c, M)-convex if for any $X \in U$, $A = (\partial_i \mathscr{D}_j V) - cId \geq 0$, in $M_n(C^{\otimes_{hc} 2}$ with $C = C_c^0(A, U : B)$, in the sense of one of the following equivalent assertions

1. $A = A^* \in M_n(C^{\otimes_{hc} 2})$ *has a semigroup of contraction* e^{-At}
2. $A = A^* \in M_n(C^{\otimes_{hc} 2})$ *has a resolvent familly for all* $\alpha > 0$, $\alpha + A$ *is invertible in* $M_n(C^{\otimes_{hc} 2})$ *and* $\|\frac{\alpha}{\alpha + A}\| \leq 1$.

3.1 Notions of non-commutative convexity and consequences

The following result is similar to the result of [GuionnetShlyakhtenkoGAFA] for (c,M) convexity.

Proposition

Assume $V \in C_c^2(A, M : B)$, *is generalized* (c, M)*-convex and assume there exists* $X^V = (X_1^V, \ldots, X_n^V)$ *satisfying Schwinger Dyson* (SD_V) *with potential* V, *with* $\|X_i^V\| \leq M/3$. *Then for any* $X = (X_1, \ldots, X_n)$, *with* $\|X_i\| \leq M/3$, *the SDE*

$$X_t = X + S_t - \int_0^t \frac{1}{2} \mathscr{D}V(X_u)du$$

has a unique globally defined solution such that $\|X_t^V - X_t\| \leq e^{-ct/2}\|X^V - X\|$. *Especially the solution of* SD_V *is unique.*

Let $A^n_{M,V,conj}$ the subset of the product of n open balls of radius M in A having conjugates variables and such that

$$\|X_t(X_0)\| < M.$$

Proposition

Under the hypothesis of our previous proposition with $V \in C^{k+2}_c(A, U : B)$, $X_t(X_0, \{S_s, s \in [0,t]\}) \in C^{k,k}_{tr,V,c}(A, U : \mathscr{B} : \mathscr{S})$. *Moreover, there exists a finite constant* C *such that :*

$$(\|X_t\|_{k,k,A^n_{M,V,conj}} - \|X_t\|_{0,0,A^n_{M,V,conj}}) \leq Ce^{-ct/2}.$$

Finally the map φ^V_t *defined, for* $P \in C^{k,k}_{tr,V,c}(A, A^n_{M,V,conj} : B)$ *by*

$$(\varphi^V_t(P))(X_0) = \tau(P(X_t)|X_0),$$

defines a semigoup there.

Proposition

Let $V_\alpha = \alpha W + (1-\alpha)V$ satisfy the hypothesis of our previous proposition for any $\alpha \in [0,1]$ and $V_\alpha \in C_c^6(A, M : B)$ (generalized (c,M) convex. Let

$$g_\alpha = \frac{1}{2}\int_0^\infty [\varphi_t^\alpha(W) - \tau(\varphi_t^\alpha(W))]dt \in C^{4,4}_{tr,V_\alpha}(A, A^n_{M,V_\alpha,conj}).$$

Then g_α satisfies the equation: $\Delta_{V_\alpha} g_\alpha = (W - \tau_{V_\alpha}(W))$. Moreover the differential equation

$$\frac{d}{d\alpha}F_\alpha = \mathscr{D}g_\alpha(F_\alpha) = (\mathscr{D}_1 g_\alpha(F_\alpha), ..., \mathscr{D}_n g_\alpha(F_\alpha))$$

has a unique solution with the initial condition $F_0 = X$ on a small time $[0, \alpha_0]$ and can be extended to $[0, \alpha + \alpha_0[$ as long as $F_\beta(X) \in A^n_{M,V_\beta,conj}, \forall \beta \in [0, \alpha[$.

3.3 Construction of transport maps.

Proposition

With the assumptions above, $F_\beta(X_1, ..., X_n)$ has law τ_{V_β} for any $\beta \in [0, \alpha + \alpha_0[$ if $(X_1, ..., X_n)$ has law τ_V. Especially $C^*(\tau_V) \simeq C^*(\tau_W), W^*(\tau_V) \simeq W^*(\tau_W)$

The key lemma is as follows :

Lemma

For F_α constructed above, then $U_\alpha = \mathscr{J}^*_{F_\alpha}(1 \otimes 1) - \mathscr{D}V_\alpha(F_\alpha)$ exists and satisfies the differential equation in L^∞

$$\frac{d}{d\alpha} U_\alpha = -\mathscr{J}\mathscr{D}g_\alpha \# U_\alpha - [d\mathscr{D}g_\alpha(\tau_{F_\alpha}).(U_\alpha)](F_\alpha).$$

As a consequence, if $U_0 = 0$, then $U_\alpha = 0, \forall \alpha \in [0, 1]$.

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Conclusion

1. The previous considerations can be applied to some cases relative to a subalgebra D, i.e. when we consider the Schwinger-Dyson equation :

$$\tau((1 \otimes E_D)(\partial_i P)) = \tau((D_i V)P) \ \forall P \in B\langle X_1, ..., X_n \rangle.$$

2. The general theory of C^k maps works well if one uses $D' \cap M^{\otimes_{ehD} n}$ with the extended Haagerup product when D, M von Neumann algebras studied by [Magajna]. At the end one needs a strong assumption on $D \subset B$ for instance valid when B is a crossed product of a trace preserving action of a countable discrete group Γ on D.
3. One of my motivations is to transport free brownian motions (relative to D in presence of a initial condition algebra B) to (weak) solutions of SDEs. (free version of Feyel-Usthunel)
4. The next step is to try solving really Monge -ampere equation for convex potentials, and then beyond the convex case. Again, the key step should be the study of a linearized pb.

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