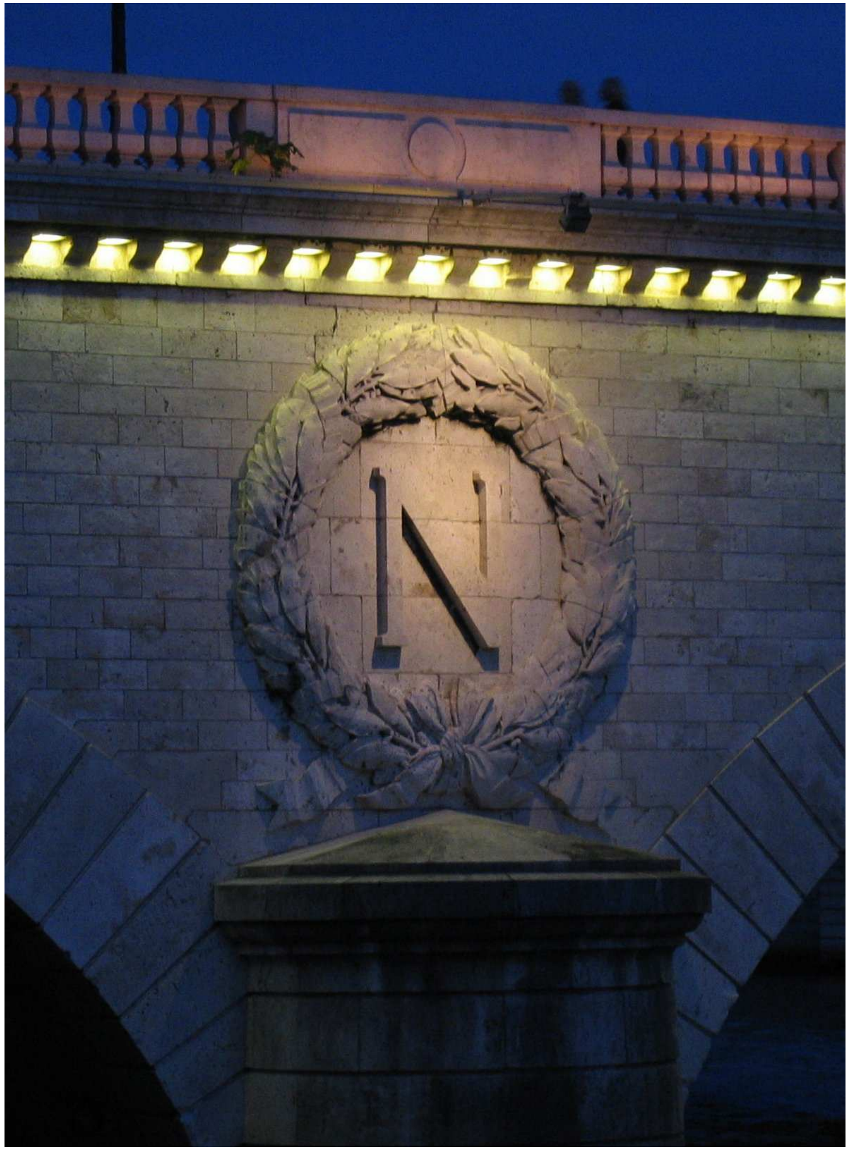
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Introduction to Random Matrices from a physicist's perspective

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Berkeley March 2007

Random matrices: a major theme in theoretical physics since Wigner (1951);

a major theme in mathematical physics and mathematics for the last 20 years.

Plan of talk

- A short history of the subject
  - Random systems, from nuclear Hamiltonians to mesoscopic systems to financial markets. . .
  - Large N limit of $U(N)$ gauge theory $\Rightarrow$ "Topological expansion" and the counting of maps and other "planar" objects $\Rightarrow$ Stat mech models on "random lattices"
  - Double scaling limit and 2D quantum gravity
  - QCD, Dijkgraaf-Vafa, etc
- Feynman diagrams and large N limit
- Counting of maps or triangulations (cf Edouard Maurel-Segala's talk)
- Computational methods: saddle point; [loop equations (cf EMS)]; [orthogonal polynomials (cf Paul Zinn-Justin)]

A short history of the subject

1. Random systems [Wigner 1951]

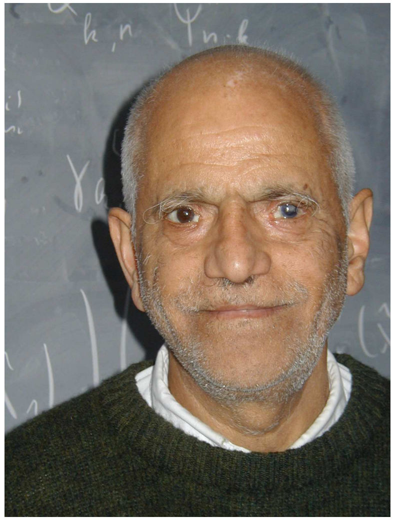
Study of spectrum of large size Hamiltonians of big nuclei, regarded as Gaussian random matrices, subject to some symmetry or reality property (GOE, GUE, GSE, ...)

Eugene P. Wigner

1902 - 1995

Freeman Dyson

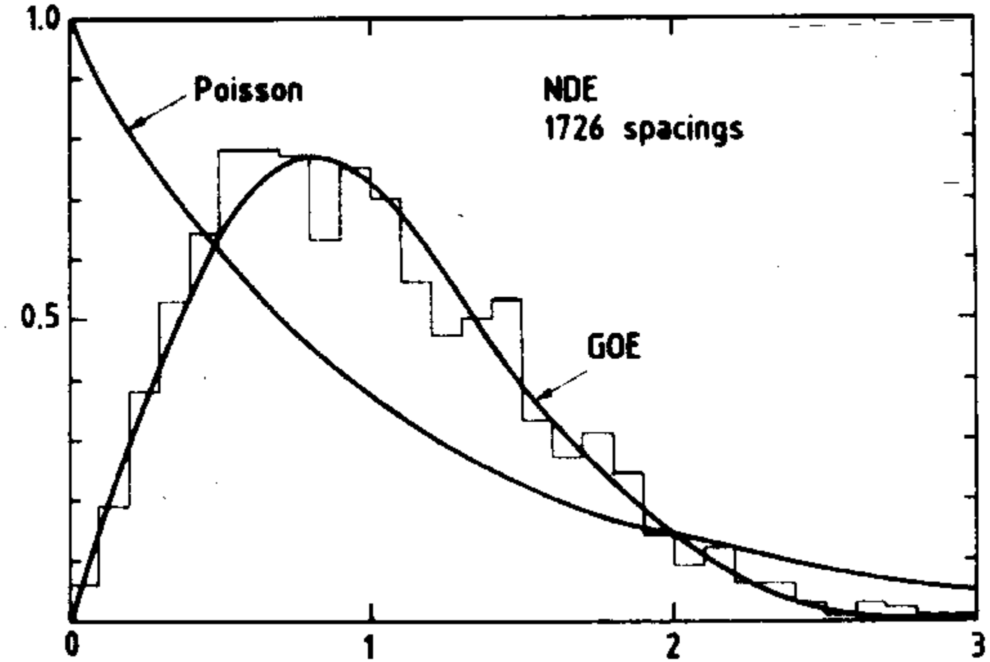
1923 -



Madan Lal Mehta

1932 - 2006

Statistics of energy levels of random Hamiltonians



Level spacing histogram for a large set of nuclear levels

("Nuclear Data Ensemble", [O. Bohigas et al]).

Other random systems and random matrices : from transport properties (for ex. universal fluctuations of electric conductivity) in disordered mesoscopic systems to financial markets. . .

2. Large N limit of $U(N)$ gauge theories

Gauge theories : quantum field theories based on a principal fibre bundle with a compact group G. "Gauge field" A (connection) lives in Lie algebra of G. For $G = U(N)$, A is a $N \times N$ (anti-Hermitian) matrix.
In the search of a non-trivial approximation, it is natural to look at large N limit, expansion parameter $1/N^2$, ['t Hooft 1974], see below.

Indeed, major simplification in the large N limit of Feynman expansions of matrix field theories. . .

Consider toy field theories : integrals over a finite number of (large size) matrices.

Basics of Feynman diagrams

Consider a Gaussian integral over n real variables x_i, $A = A^T > 0$ def. matrix

$$\int d^n x\, e^{-\frac{1}{2}\sum x_i A_{ij} x_j} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A}$$

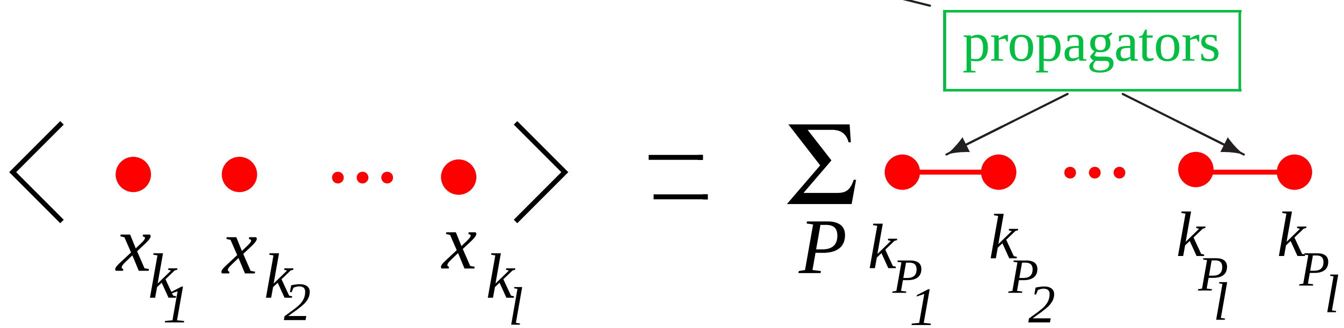
$$\int d^n x\, \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j + \sum b_i x_i} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A}\, \mathrm{e}^{\frac{1}{2}\sum b_i A^{-1}_{ij} b_j}$$

Differentiate w.r.t. b_i

$$\langle x_{k_1} x_{k_2} \cdots x_{k_\ell} \rangle := \frac{\int d^n x\, x_{k_1} x_{k_2} \cdots x_{k_\ell} e^{-\frac{1}{2} x.A.x}}{\int d^n x\, e^{-\frac{1}{2} x.A.x}} = \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathrm{e}^{\frac{1}{2} b.A^{-1}.b}\Big|_{b=0}$$

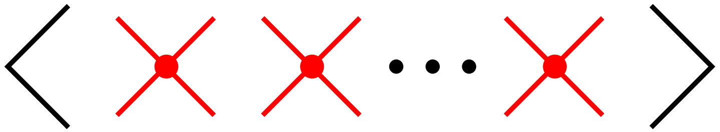
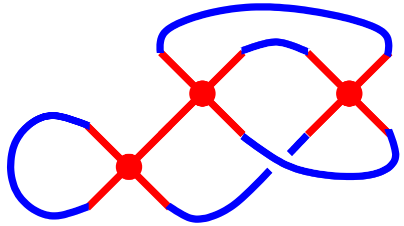
$$= \sum_{\substack{\text{all distinct} \\ \text{pairings } P \text{ of the } k}} A^{-1}_{k_{P_1} k_{P_2}} \cdots A^{-1}_{k_{P_{\ell-1}} k_{P_\ell}}$$

Wick theorem



Wick theorem also applies to monomials ($n = 1$ variable for simplicity):

p vertices   propagator A^{-1}

$\langle (x^4)^p \rangle =$  $= \sum_{\text{graphs}}$ 

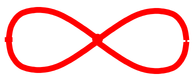
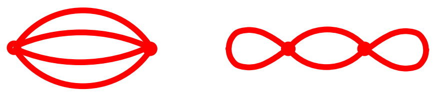
Non Gaussian integrals ($g < 0$): power series "perturbative" expansions

$$Z = \int dx\, \mathrm{e}^{-\frac{1}{2}Ax^2 + \frac{g}{4!}x^4} = \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{g^p}{p!} \int dx \left(\frac{x^4}{4!}\right)^p \mathrm{e}^{-\frac{1}{2}Ax^2}$$

$$= \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \sum_{\substack{\text{graphs } \mathcal{G} \text{ with } 2p \text{ lines} \\ \text{and } p\ 4\text{-valent vertices}}} \frac{g^p}{|\mathrm{Aut}\,\mathcal{G}|} A^{-2p}$$

$\log Z$ = <u>connected</u> Feynman diagrams

$$= \frac{g}{8A^2} + \frac{g^2}{A^4}\left(\frac{1}{2.4!} + \frac{1}{2^4}\right) + \cdots$$

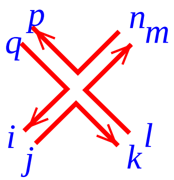
$=$   $+ \cdots$

Matrix Integrals: Feynman Rules

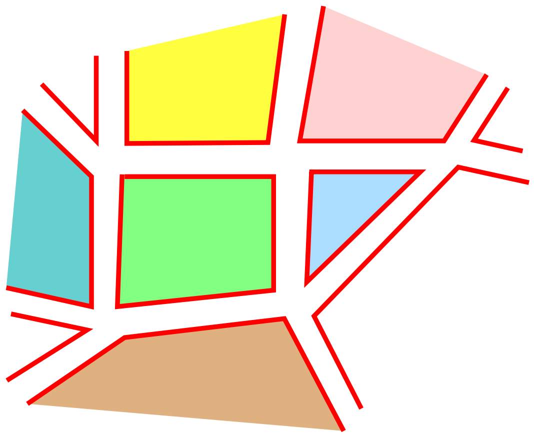
$N \times N$ Hermitean matrices M, $dM = \prod_i dM_{ii} \prod_{i<j} d\Re e M_{ij}\, d\Im m M_{ij}$

$$Z =: e^F = \int dM\, \mathrm{e}^{N[-\frac{1}{2}\mathrm{tr}\, M^2 + \frac{g}{4}\mathrm{tr}\, M^4]}$$

Feynman rules: propagator  $= \frac{1}{N}\delta_{i\ell}\delta_{jk}$ ['t Hooft]

4-valent vertex :  $= gN\delta_{jk}\delta_{\ell m}\delta_{np}\delta_{qi}$

For each connected diagram contributing to $\log Z$: fill each closed index loop with a disk $\Rightarrow$ discretized closed 2-surface Σ

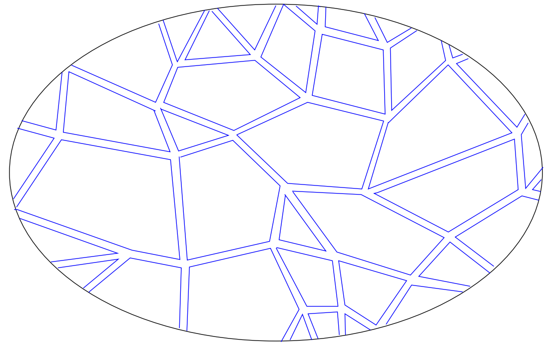


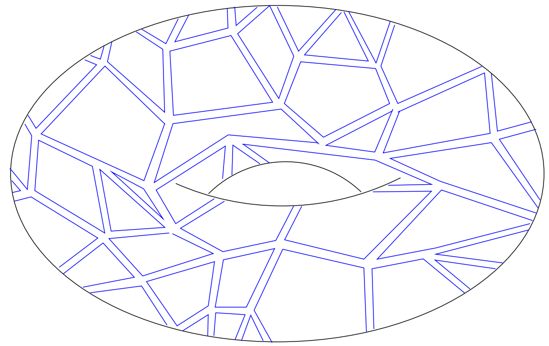
Power of N in a connected diagram

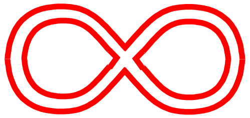
- each vertex $\longrightarrow N$;
- each double line $\longrightarrow N^{-1}$;
- each loop $\longrightarrow N$.

Thus $N^{\#\text{vert.}-\#\text{lines}+\#\text{loops}} = N^{\chi_{\text{Euler}}(\Sigma)}$

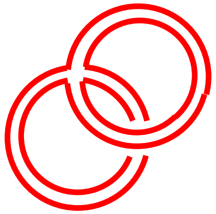
['t Hooft (1974)]. For example, compare







gN^2



gN^0

A topological expansion :

$$F = \log Z = \sum_{\text{conn. surf.}\Sigma} N^{2-2\text{genus}(\Sigma)} \frac{g^{\#\text{vert.}(\Sigma)}}{\text{symm. factor}}$$

$$= \sum_{h=0}^{\infty} N^{2-2h} F^{(h)}(g).$$

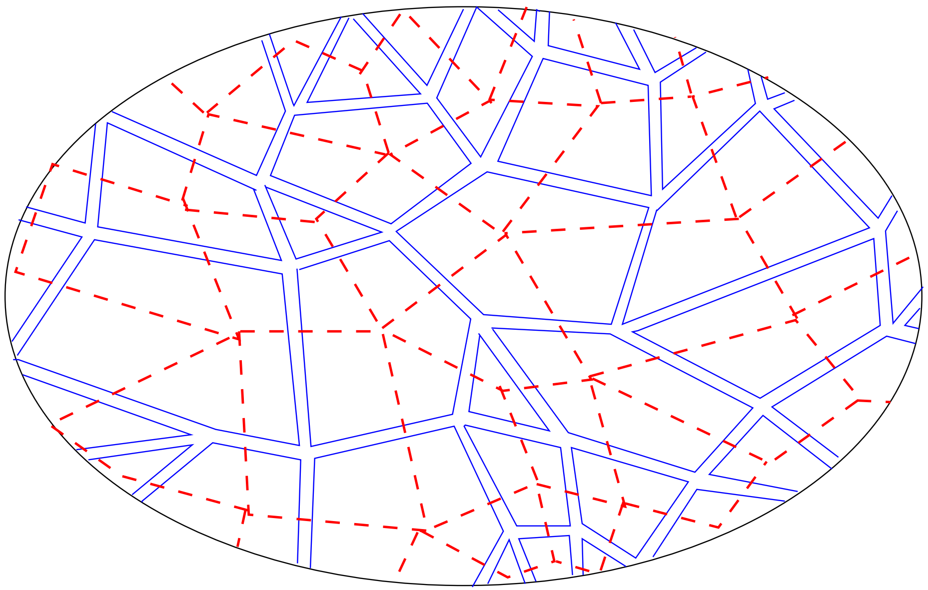
Thus large N limit of matrix integral $\int DMe^{-N\mathrm{tr}\,(M^2+\frac{g}{4}M^4)}$ = generating function of planar 4-valent graphs...(cf census of planar maps by Tutte)

[Brézin, Itzykson, Parisi, Z. 1978]

$$\lim_{N\to\infty}\frac{1}{N^2}\log Z = \sum_{\substack{\text{planar diagrams}\\ \text{with } n \text{ 4-vertices}}} \frac{g^n}{\text{symm.factor}}$$

or in a dual way, of *quadrangulations* of 2D surfaces of genus 0

[Kazakov; David; Kazakov-Kostov-Migdal; Ambjørn-Durhuus-Fröhlich '85]

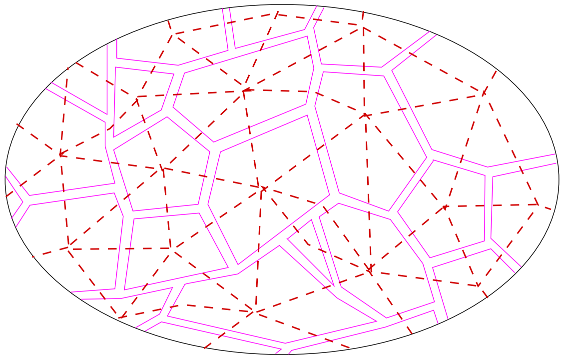


Thus large N limit of matrix integral $\int DMe^{-N\mathrm{tr}\,(M^2+\frac{g}{3}M^3)}$ = generating function of planar 3-valent graphs. . . [Brézin, Itzykson, Parisi, Z. 1978]

or in a dual way, of *triangulations* of 2D surfaces of genus 0

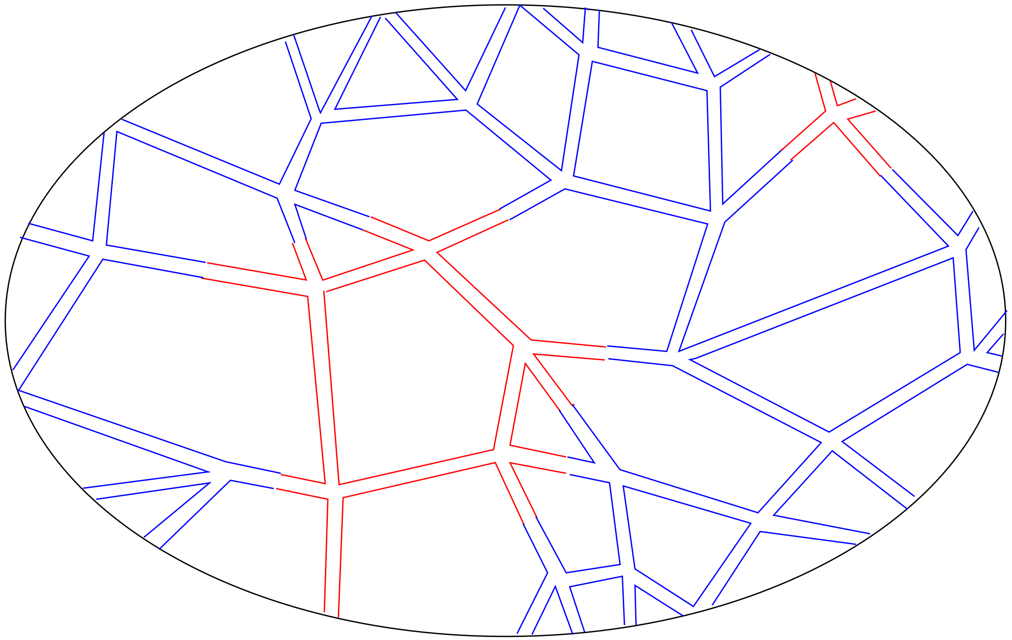
[Kazakov; David; Kazakov-Kostov-Migdal; Ambjørn-Durhuus-Fröhlich '85]

Triangulated surfaces and discrete 2D gravity



Thus : Large N limit of matrix integrals $\Rightarrow$ Counting of planar objects : maps, triangulations, "alternating" knots and links [P Z-J & J-B Z], etc, or of objects of higher topology . . .

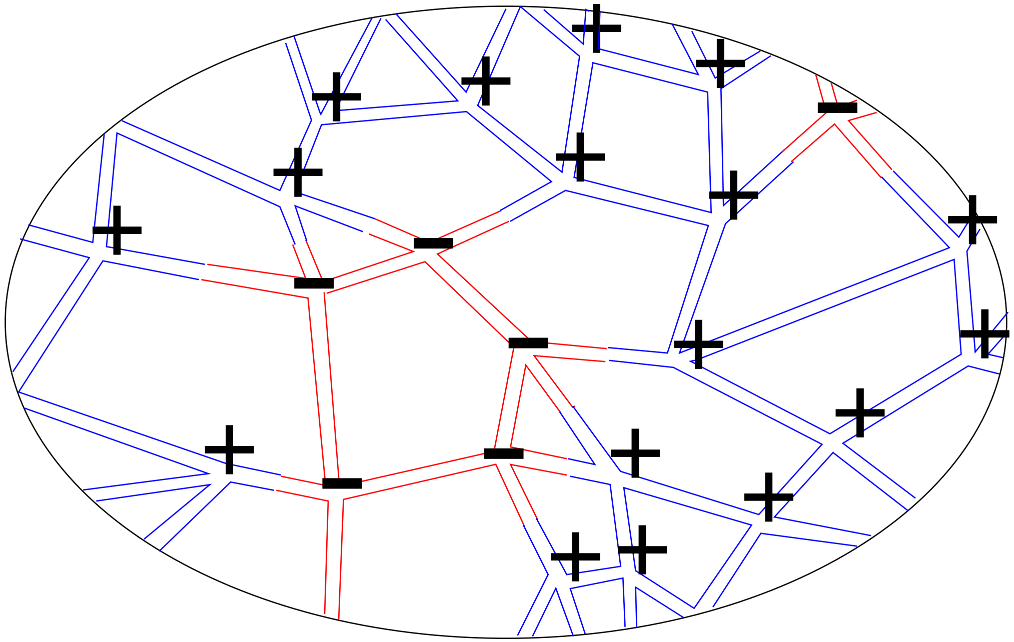
Large N limit of matrix integral $\int DADBe^{-N\mathrm{tr}(A^2+cAB+B^2+gA^4+gB^4)} =$ generating function of bicolored planar 4-valent graphs...



Large N limit of matrix integral $\int DADB e^{-N \mathrm{tr}(A^2+cAB+B^2+gA^4+gB^4)} =$ generating function of bicolored planar 4-valent graphs. . .

i.e. describes Ising model on a random quadrangulated sphere

[Boulatov-Kazakov 1985]



etc etc, many variations on that theme

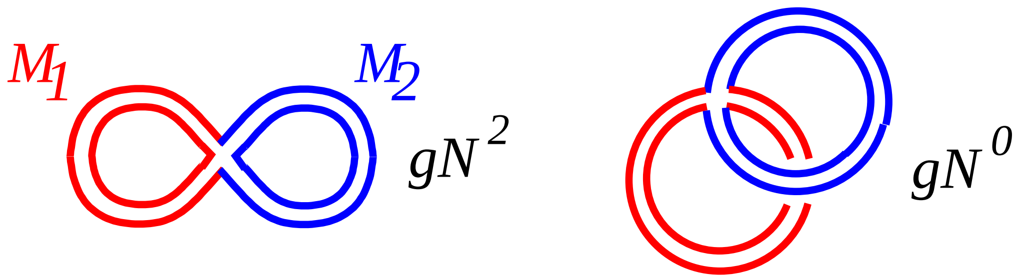
“Statistical mechanics models on a random lattice”

Two remarks useful in connection with free probabilities...

- Factorization property $\langle \frac{1}{N}\mathrm{tr}\, P_1 \frac{1}{N}\mathrm{tr}\, P_2 \rangle = \underbrace{\langle \frac{1}{N}\mathrm{tr}\, P_1 \rangle \langle \frac{1}{N}\mathrm{tr}\, P_2 \rangle}_{\text{disconnected diagrams}} + O(\frac{1}{N^2})$

- Compare (in Gaussian theory $e^{-\mathrm{tr}\,(M_1^2+M_2^2)}$)

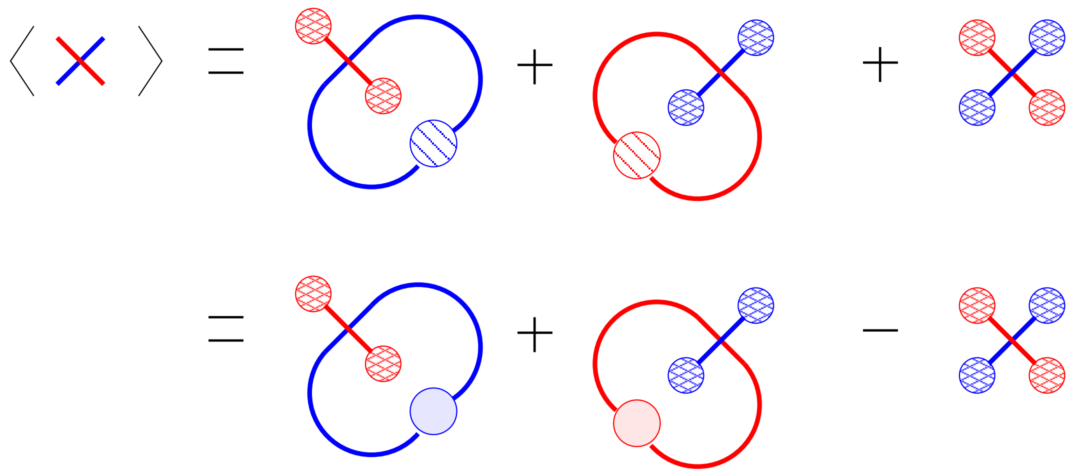
$$\langle \frac{1}{N}\mathrm{tr}\, M_1^2 M_2^2 \rangle \qquad \text{and} \qquad \langle \frac{1}{N}\mathrm{tr}\, M_1 M_2 M_1 M_2)\rangle$$

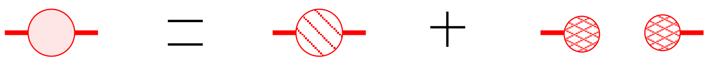


Similar behavior in non Gaussian theory $e^{-\mathrm{tr}\,(V_1(M_1)+V_2(M_2))}$ provided $\langle M_1 \rangle = 0$, $\langle M_2 \rangle = 0$.

If $\langle M_1 \rangle \neq 0$, $\langle M_2 \rangle \neq 0$,

$\langle \mathrm{tr}\, M_1 M_2 M_1 M_2 \rangle =$





Compare $\tau(a_1 a_2 a_1 a_2) = \tau(a_1^2)\tau^2(a_2) + \tau^2(a_1)\tau(a_2^2) - \tau^2(a_1)\tau^2(a_2)$ for two free variables

3. Double scaling limit

[Brézin-Kazakov; Douglas-Shenker; Gross-Migdal 1989, ...]
$F^{(0)}(g)$ and more generally $F^{(h)}(g)$ have a singularity at $g = g_c$,

$$F^{(h)}(g) \sim (g_c - g)^{(2-\gamma_{\rm str})(1-h)}$$

with $\gamma_{\rm str}$, the "string susceptibility", typically equal to $-1/2$ (for the simplest models M^3 or M^4 above), see below.

As $g \to g_c$, $\langle\#\text{ triangles}\rangle = \frac{\partial \log F}{\partial g}$ diverges. Expect to make contact with *continuum* 2D gravity. Keep all genera h in $\sum_{h=0} N^{2-2h} F^{(h)}(g)$ by letting $g_c - g \to 0$ as $N \to \infty$ in such a way that $(g_c - g)^{(2-\gamma_{\rm str})/2} N = \kappa$ fixed.

Very interesting limit : appearance of integrable equations (KdV ...), solutions to Painlevé equations ...

Thus, double scaling limit $\Rightarrow$ models of 2D quantum gravity

4. Other physical applications

• Cell decomposition of moduli space of Riemann surfaces, intersection numbers . . .
[Witten, Kontsevich, 1991 . . .]

• QCD, the Dirac operator $\not{D} = \not{\partial} + \not{A}$ in the presence of a gauge field and RMT [Verbaarschot et al.]

• Dijkgraaf-Vafa 2002 : computing the effective action of supersymmetric gauge theories in terms of matrix integrals

etc etc

Computational techniques

Consider integral over $N \times N$ Hermitian matrices

$$Z = \int dM e^{-N \mathrm{tr}\, V(M)} ,$$

$V(M)$ a polynomial of degree $d+1$. For ex. $V_3(M) = (\frac{1}{2}M^2 + \frac{g}{3}M^3)$ and $V_4(M) = (\frac{1}{2}M^2 + \frac{g}{4}M^4)$. Note that multi-traces are excluded, for example $(\mathrm{tr}\, M^2)^2$.

Integrand and measure are invariant under $U(N)$ transformations $M \longrightarrow UMU^\dagger$. Express both in terms of *eigenvalues* $\lambda_1, \cdots, \lambda_N$ of M :

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N \sum_{i=1}^{N} V(\lambda_i)} ,$$

Several ways to treat this integral: saddle point approximation ; orthogonal polynomials; “loop equation”. . .

1. Saddle point approximation

Rewrite

$$Z = \int \prod_{i=1}^{N} d\lambda_i \exp\left(2\sum_{i<j} \log|\lambda_i - \lambda_j| - N\sum_{i=1}^{N} V(\lambda_i) \right)$$

In the large N limit, if $\lambda \sim O(1)$, both terms in exponential are of order N^2. Look for the stationary point, i.e. the solution of

$$\frac{2}{N}\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = V'(\lambda_i) \ . \qquad (*)$$

To solve this problem, introduce the resolvent

$$G(x) = \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{x-M} \right\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N} \frac{1}{x-\lambda_i} \right\rangle \ .$$

Computing its square leads after some algebra to

$$G^2(x) = \frac{1}{N^2}\left\langle \sum_{i,j=1,\cdots,N} \frac{1}{(x-\lambda_i)(x-\lambda_j)} \right\rangle = \cdots = -\frac{1}{N}G'(x) + V'(x)G(x) - P(x)$$

with $P(x) := \frac{1}{N}\left\langle \sum_{i=1}^{N} \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle$ a *polynomial* in x of degree $d-1$, *i.e.*

$$G^2(x) - V'(x)G(x) + \frac{1}{N}G'(x) + P(x) = 0 \ .$$

(Beware ! Not exact for N finite!) For N large, neglect the $1/N$ term $\Rightarrow$ quadratic equation for $G(x)$, with yet unknown polynomial P, hence

$$G(x) = \frac{1}{2}\left(V'(x) - \sqrt{V'(x)^2 - 4P(x)}\right)$$

(minus sign in front of $\sqrt{}$ dictated by the requirement that for large $|x|$, $G(x) \sim 1/x$.)

In that large N limit, the λ's form a continuous distribution with density

$\rho(\lambda)$ on a support S , $\int_S d\lambda\rho(\lambda) = 1$, and $G(x) = \int_{-2a'}^{2a''} \frac{d\mu\rho(\mu)}{x-\mu}$.
For a purely Gaussian potential $V(\lambda) = \frac{1}{2}\lambda^2$, Wigner's "semi-circle law":
$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$ on the segment $\lambda \in [-2,2]$.
For more general potentials, assume first S to be still a finite segment $[-2a',2a'']$, in such a way that (*) becomes

$$2\,\text{P.P.} \int_{-2a'}^{2a''} \frac{d\mu\rho(\mu)}{\lambda-\mu} = V'(\lambda) \qquad \text{if } \lambda \in [-2a',2a'']\,.$$

(P.P.= principal part), expressing that, along its cut,

$$G(x \pm i\varepsilon) = \frac{1}{2}V'(\lambda) \mp i\pi\rho(x) \qquad x \in [-2a',2a'']\,.$$

Thus

$$G(x) = \frac{1}{2}V'(x) - Q(x)\sqrt{(x+2a')(x-2a'')}$$

where the coefficients of the polynomial $Q(x)$ and a', a'' are determined by the condition that $G(x) \sim 1/x$ for large $|x|$. Q is of degree $d-1$. The solution is unique (under the one-cut assumption).

Example For the quartic potential $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, by symmetry $a' = a'' =: a$,

$$G(x) = \frac{1}{2}(x + gx^3) - (\frac{1}{2} + \frac{g}{2}x^2 + ga^2)\sqrt{x^2 - 4a^2}$$

with a^2 the solution of

$$3ga^4 + a^2 - 1 = 0 \qquad (EQa^2)$$

which goes to 1 as $g \to 0$ (a limit where we recover Wigner's semi-circle law). From this we extract

$$\rho(\lambda) = \frac{1}{\pi}(\frac{1}{2} + \frac{g}{2}\lambda^2 + ga^2)\sqrt{4a^2 - \lambda^2}$$

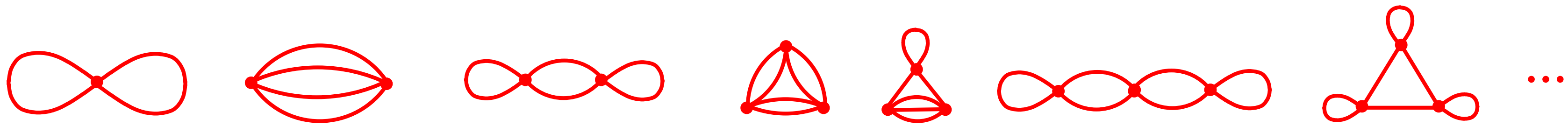
and we may compute all invariant quantities like the free energy or the moments

$$G_{2p} := \left\langle \frac{1}{N}\mathrm{tr}\, M^{2p} \right\rangle = \int d\lambda\, \lambda^{2p}\rho(\lambda)\ .$$

For example $G_2 = (4 - a^2)a^2/3$, $G_4 = (3 - a^2)a^4$, etc. All these functions of a^2 are singular as functions of g at the point $g_c = -\frac{1}{12}$ where the two roots of (EQa^2) coalesce. For example the genus 0 free energy

$$F^{(0)}(g) := \lim_{N\to\infty}(1/N^2)\log\left(\frac{Z(g)}{Z(0)}\right) = \frac{1}{2}\log a^2 - \frac{1}{24}(a^2-1)(9-a^2)$$

$$= \sum_{p=1}(\frac{3g}{t^2})^p \frac{(2p-1)!}{p!(p+2)!} \qquad [\text{Tutte 62, BIPZ 78}]$$



has a power-law singularity

$$F^{(0)}(g) \underset{g\to g_c}{\approx} |g-g_c|^{5/2}$$

which reflects on its series expansion

$$F^{(0)}(g) = \sum_{n=0}^{\infty} f_n g^n \quad , \qquad f_n \underset{n\to\infty}{\approx} \text{const}|g_c|^{-n} n^{-7/2} \,.$$

Comments

i) Nature of the $1/N^2$ and of the g expansions, algebraic singularity at finite g_c

ii) “Universal” singular behavior at g_c

iii) Extension to several cuts, the rôle of the algebraic curve (cf Eynard).

iv) Connected correlation functions and “free (or non crossing) cumulants”

v) Factorization property, localization of the matrix integral and the “master field” [. . .]

2. Orthogonal polynomials

$$\int d\lambda P_m(\lambda)P_n(\lambda)e^{-NV(\lambda)} = h_n\delta_{mn}$$

[Mehta, Bessis, . . .]

cf Paul Z-J . . .

3. Loop (or Schwinger-Dyson) equations

$$\int dM \frac{\partial}{\partial M_{ij}}\{\cdots e^{-N\mathrm{tr}V(M)}\} = 0$$

and make use of factorization property . . .

cf Edouard M-S . . .

Comments on “Free” or “non-crossing” cumulants f_n

[BIPZ 78, Cvitanovic 81, Voiculescu 85, Speicher 94, Biane, PZ-J]

Generating function of moments $m_n = \frac{1}{N}\operatorname{tr} M^n$

$$Z(j) = \langle \frac{1}{N}\operatorname{tr}\frac{1}{1-jM}\rangle = \sum_{n=0}^{\infty} j^n m_n$$

or $G(u) = u^{-1}Z(u^{-1}) = \sum_{n=0}^{\infty} u^{-n-1} m_n$. The generating function of free cumulants

$$W(z) = 1 + \sum_{n=1}^{\infty} z^n f_n = 1 + \widetilde{W}(z)$$

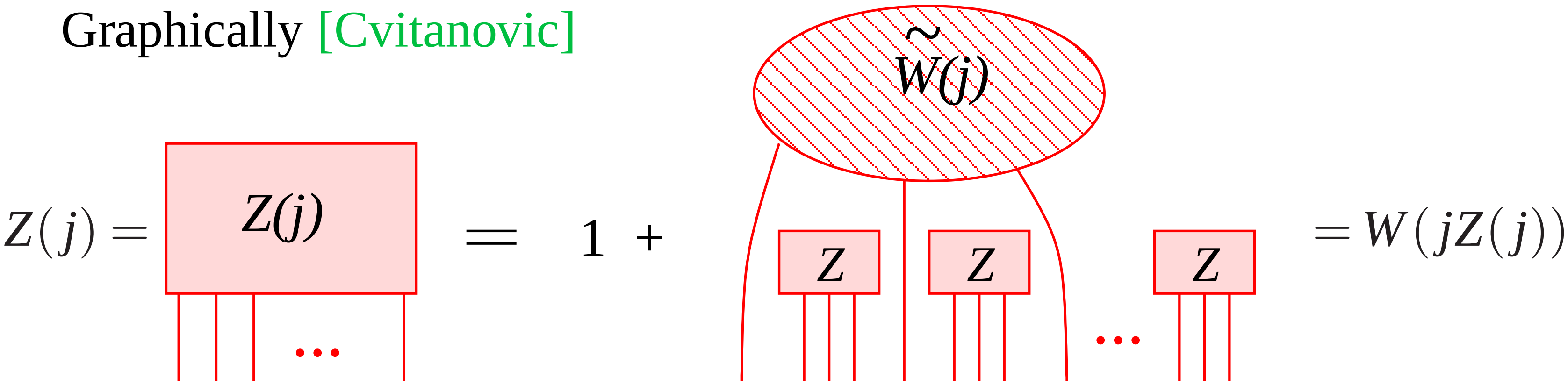
or $P(z) = z^{-1}W(z)$, is defined by the relations

$$W(z) = Z(j(z)) \qquad \text{with} \quad j(z) = z/W(z)$$

or equivalently

$$Z(j) = W(z(j)) \qquad \text{with} \quad z(j) = jZ(j)$$

Graphically [Cvitanovic]



These relations amount to saying that P et G are functional inverses of one another $P \circ G(u) = u$. Indeed
$P(G(u)) = G^{-1}(u)W(G(u)) = uZ^{-1}(u^{-1})W(u^{-1}Z(u^{-1})) = u$, since
$Z(u^{-1}) = W(u^{-1}Z(u^{-1}))$.

Using Lagrange formula, one computes

$$m_k = \sum_{\alpha \vdash k} \frac{k!}{(k+1-\sum \alpha_q)!} \frac{f_1^{\alpha_1}}{\alpha_1!} \frac{f_2^{\alpha_2}}{\alpha_2!} \cdots$$

or conversely $f_k = -\sum_{\alpha \vdash k} \frac{(k-2+\sum \alpha_q)!}{k!} \frac{(-m_1)^{\alpha_1}}{\alpha_1!} \frac{(-m_2)^{\alpha_2}}{\alpha_2!} \cdots$

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