Dirichlet Forms on C^*-algebras

A Review

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Berkeley, march 25, 2007

Part 1.

Dirichlet Forms.

Data (60's)

A is a C*-algebra, τ l.s.c. semifinite faithful trace on A
$\big(\tau(a^*a) = \tau(aa^*)\big)$;

$L^2(A,\tau)$ = G.N.S. space = completion of
$\mathcal{N}_\tau = \{a \in A \,/\, \tau(a^*a) < +\infty\} \ \big(= A \cap L^2(A,\tau)\big)$
$\Lambda_\tau : \mathcal{N}_\tau \to L^2(A,\tau)$ tautological embedding;

$J : L^2(A,\tau) \to L^2(A,\tau)$ canonical antilinear isometric involution: $J(\Lambda_\tau(a)) = \Lambda_\tau(a^*)$.

$\pi_\tau : A \to \mathcal{L}(L^2(A,\tau))$ left action of A on $L^2(A,\tau)$:
$\pi_\tau(a)\Lambda_\tau(b) = \Lambda_\tau(ab)$
shortly denoted $\pi_\tau(a)\xi = a\xi$;

$\pi_\tau(A)''$= von Neumann algebra generated by A in $\mathcal{L}(L^2(A,\tau))$.

Fact: $\pi_\tau(A)' = J\pi_\tau(A)''J$.

Functional calculus in $L^2(A,\tau)$:

$$\xi \in L^2(A,\tau),\ \xi = J\xi:\ \xi = \int_{-\infty}^{+\infty} \lambda dE(\lambda),$$

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\xi) = \int_{-\infty}^{+\infty} f(\lambda)dE(\lambda).$$

Symmetric semigroups

Semigroup means *Markov semigroup*, i.e

σ-weakly pointwise continuous semigroup of completely positive contractions of $\pi_\tau(A)''$.

Symmetric semigroup = semigroup symmetric/τ :

$$\tau(b\,\varphi_t(a)) = \tau(\varphi_t(b)a)\,,\ \forall a, b \in \pi_\tau(A)''_+\,,\ \forall t \in \mathbb{R}_+\,.$$

L^2-generator :

$\{\varphi_t\}_{t\in\mathbb{R}_+}$ extends as a semigroup of s.a. contractions in $L^2(A,\tau)$ $\rightarrow$ (Hille-Yoshida)

$$\varphi_t = e^{-tL}$$

L self-adjoint densely defined nonnegative in $L^2(A,\tau)$.

N.B.

$\{\varphi_t\}$ acts in the von Neumann algebra.

L acts in the Hilbert space $L^2(A,\tau)$.

Note the $-$ sign in definition of L.

Dirichlet forms

E closed densely defined nonnegative quadratic form on $L^2(A, \tau)$.

Definition. E is a *Dirichlet form* if one of the two equivalent properties hold:

(a) $\forall \xi \in L^2(A, \tau)$ s.t. $\xi = J\xi$, $E[\xi \wedge 1] \leq E[\xi]$.

(b) $\forall \xi \in L^2(A, \tau)$ s.t. $\xi = J\xi$,
$\forall f \in C^1(\mathbb{R})$ with $f(0) = 0$, one has
$E[f(\xi)] \leq ||f||_{C^1}^2 \, E[\xi]$, with $||f||_{C^1} = \sup\{|f'(t)| \,/\, t \in sp(\xi)\}$.

E is a *complete Dirichlet form* if, $\forall n \geq 1$,

$$E_n : \; E_n[(\xi_{ij})] = \sum_{i,j} E_n[\xi_{ij}]$$

is a Dirichlet form on $L^2(A \otimes M_n(\mathbb{C}), \tau \otimes \tau_n)$.

Basic Theorem.

Complete Dirichlet forms $\longleftrightarrow$ *symmetric semigroups*

Beurling/Deny (59), Albeverio/Hoegh-Krohn (77), S. (90), Lindsay-Davies (92), Cipriani (97), etc.

Correspondance :

From the semigroup to the Dirichlet form :

$$\varphi_t = e^{-tL} \text{ in } L^2(A,\tau) \longrightarrow E[\xi] = ||L^{1/2}\xi||^2_{L^2} .$$

From the Dirichlet form to the semigroup :

1/ Resolvents on $M = \pi_\tau(A)''$: show

$$\frac{\lambda I}{\lambda I + L} : M \cap L^2(A,\tau) \to M \cap L^2(A,\tau) , \ \forall \lambda \in \mathbb{R}^*_+$$

+ (complete) positivity on M + contractivity on M.

2/ Family of resolvents $\rightarrow$ semigroup by Trotter-Kato formula.

Part 2.

Associated differential calculus.

Definitions.

1. *Dirichlet algebra:* $\mathcal{B} = A \bigcap \mathcal{D}(E)$.

Proposition. [LD] E complete Dirichlet form. Then *$\mathcal{B}$ is a $*$-subalgebra of A.*

2. *Regular Dirichlet form:* $\mathcal{B}$ is $\left\{ \begin{array}{c} \text{dense in } A \\ \text{a core for } E\,. \end{array} \right.$

3. *Symetric differential calculus:*

- $\mathcal{H}$ a $A - A$-bimodule (possibly degenerate),
- $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ antilinear involution,

$$\mathcal{J}(a\xi b) = b^* \mathcal{J}(\xi) a^* \,,$$

- $B \subset A \cap L^2(A, \tau)$ dense $*$-algebra,
- $\partial : \ B \to \mathcal{H}$ a symmetric derivation:

$$\partial(a^*) = \mathcal{J}(\partial(a))\,, \ a \in B\,,$$

$$\partial(ab) = a\partial(b) + \partial(a)b\,, \ a, b \in B$$

closable from $L^2(A, \tau)$ into $\mathcal{H}$.

Theorem 1. [S]

If $(\mathcal{H}, \mathcal{J}, B, \partial)$ *is a symmetric differential calculus, then the closure of*

$$L^2(A,\tau) \supset B \to E[b] = ||\, \partial(b)\, ||^2_{\mathcal{H}}$$

is a regular complete Dirichlet form on $L^2(A,\tau)$ *.*

Theorem 2. [CS]

And vice versa.

i.e. if E is a regular complete Dirichlet form on $L^2(A,\tau)$, then there exists a symmetric differential calculus with $B = \mathcal{B}$ and $E[b] = ||\, \partial(b)\, ||^2_{\mathcal{H}}\,, \forall b \in \mathcal{B}\,.$

Remarks.

1/ Dirichlet forms are related to the von Neumann algebra $\pi_\tau(A)''$, while regular Dirichlet forms are related to the C*-algebra A.

2/ The left (resp. right) representation of A in $\mathcal{L}(\mathcal{H})$ can be **disjoint** form the left (resp. right) of A in $\mathcal{L}(L^2(A,\tau))$.
This happens for harmonic structures on fractals.
This does not happen if $\mathcal{D}(L) \cap A$ contains a dense $*$-subalgebra of A.

3/ The left or right representation of A in $\mathcal{L}(\mathcal{H})$ can have a degenerate part (related to killing).

Relationship with the "carré du champ"

$\mathcal{D}(L) \cap A$ needs not to contain a dense algebra,
but, for $a, b \in \mathcal{B}$,

$$\Gamma(b,a) = \frac{1}{2}\big[- L(b^*a) + b^*L(a) + L(b^*)a \big]$$

makes sense in the dual space A^*. Moreover

$\forall n$, $\forall a_1, \cdots, a_n \in \mathcal{B}$:

$$\big[\Gamma(a_j, a_i)\big]_{i,j=1..n} \in M_n(A^\circ)^*_+ \,.$$

$\mathcal{H}$ is generated by the $\partial(\alpha)a$, $\alpha \in \mathcal{B}$, $a \in A$, with scalar product

$$\big(\partial(\beta)b \,\big|\, \partial(\alpha)a\big) = \big\langle \Gamma(\beta^*, \alpha)\,,\, ab^* \big\rangle_{A^*,A} \,.$$

+ obvious right action

$$\big(\partial(\alpha)a\big)b = \partial(\alpha)(ab)\,,\ \alpha \in \mathcal{B}\,,\ a, b \in A\,,$$

+ left action (much less obvious)

$$b\big(\partial(\alpha)a\big) = \partial(b\alpha)a - \partial(b)\alpha a\,,\ \alpha, b \in \mathcal{B}\,,\ a \in A\,.$$

N.B. For defining $\Gamma(b,a)$, write

$$L = \lim_{\varepsilon \downarrow 0} L^\varepsilon = \frac{L}{I + \varepsilon L} = \frac{1}{\varepsilon}\left(I - \frac{I}{I + \varepsilon L}\right)$$

$$\Gamma^\varepsilon(b,a) = \frac{1}{2}\big[-L^\varepsilon(b^*a) + b^*L^\varepsilon(a) + L^\varepsilon(b^*)a\big] \in L^1(A,\tau) \subset A^*$$

and make $\varepsilon \downarrow 0$: $\Gamma^\varepsilon(b,a) \to \Gamma(b,a)$ weakly in A^*.

Useful both ways:

1/ Start with a differential calculus

$\longrightarrow$ Dirichlet form;

$\longrightarrow$ Markov semigroup;

$\longrightarrow$ Stochastic process.

2/ Start with a symmetric semigroup or a Dirichlet form

$\longrightarrow$ Tangent bundle ${}_A\mathcal{H}_A$ $\big(=$ "$L^2(TM)$"$\big)$

$\longrightarrow$ Gradient operator $\partial : \mathcal{B} \to \mathcal{H}$
$\mathcal{D}(\overline{\partial}) =$ "H^1(non commutative Riemannian manifold)".

$\longrightarrow$ $L = \partial^* \partial =$ divergence of the gradient.

Part 3.

Examples.

General examples.

Pure jump Dirichlet forms (classical):

$$A = C(X)\,,\ \tau = m$$

μ symmetric Radon measure on $X \times X \setminus \{\text{diagonal}\}$

$$E[f] = \int_{X\times X} |f(x) - f(y)|^2 d\mu(x,y)\,;$$

$$\mathcal{H} = L^2(X \times X, \mu)\,;$$

$$\partial f(x,y) = f(x) - f(y)\ :\ \partial f = f \otimes 1 - 1 \otimes f = \left[1_{X\times X}\,,\, f\,\right].$$

(if μ unbounded, use an approximate unit.)

Local Dirichlet forms (classical):

$$A = C(X)\,,\ \tau = m$$

Proposition. E is local iff, on the bimodule $\mathcal{H}$, the left and right actions coincide:

$$f\xi = \xi f\,,\ \forall \xi \in \mathcal{H}\,,\ \forall f \in C(X)\,.$$

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\nu(x)\,.$$

∂ ?

Negative type functions on groups :

$$A = C_r^*(G)$$

G locally compact group, d negative type function

$\varphi_t(\lambda(f)) = \lambda(e^{-td}f)$ symmetric semigroup

$$E[\lambda(f)] = \int_G d(g)|f(g)|^2 dg$$

$$\exists\, H, \,,\ \exists\, \pi : G \to U(H)\,,\ \exists\, \xi : G \ni g \to \xi(g) \in H$$

$$\xi(g_1 g_2) = \xi(g_1) + \pi(g_1)\xi(g_2)\ g_1, g_2 \in G$$

$$\text{s.t.}\quad d(g) = ||\xi(g)||_H^2\,, \quad \forall\, g \in G\,.$$

Then

$$\begin{aligned} \mathcal{H} &= {}_{\pi\otimes\lambda}H \otimes L^2(G)_{i\otimes\rho}\,, \\ &= L^2(G, H) \end{aligned}$$

$$f \in C_c(G)\ :\ \partial f = \{g \to f(g)\xi(g)\} \in L^2(G, H)\,.$$

Bounded Dirichlet form :

Any A, τ .

ψ completely positive contraction of A, symmetric/τ

$$L = I - \psi \, , \, \varphi_t = e^{-t} e^{t\psi}$$

$\mathcal{H}$ = imprimitivity bimodule of ψ :

$$\left\langle b \otimes_\psi \eta \, , \, a \otimes_\psi \xi \right\rangle_{\mathcal{H}} = \left\langle \eta \, , \, \psi(b^* a)\xi \right\rangle_{L^2(\tau)} ,$$
$$\xi, \eta \in L^2(A, \tau) \, , \; a, b \in A$$

$$\partial(a) = a \otimes_\psi 1_A - 1_A \otimes_\psi \Lambda_\tau(a)$$
$$= \left[a \, , \, 1_A \otimes_\psi 1_A\right]$$

(if A has no unit, use an approximate unit)

$\longrightarrow$ (approximately) inner derivation.

Remember : for any L,

$$L = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(I - \frac{I}{I + \varepsilon L} \right) = \lim_{e \downarrow 0} \frac{1}{\varepsilon} \big(I - \text{c.p. contraction} \big) \, .$$

$\longrightarrow \quad \mathcal{H} = \lim_{\varepsilon \downarrow 0} \mathcal{H}^\varepsilon$

with $\mathcal{H}^\varepsilon$ imprimitivity bimodule, **weakly** in the category of representations of $A \otimes_{max} A^\circ$.

Examples related to Riemannian geometry

Heat semigroup on functions: $A = C_0(M)$

M Riemannian manifold, $L = -\Delta$, $\varphi_t = e^{t\Delta}$

$$\mathcal{H} = L^2(TM)\,,\ \partial = \nabla\,,\ E(f) = \int_M |\nabla f|^2 dv\,.$$

+ other examples with perturbated Δ and measure.

Transverse heat semigroup of a Riemannian folitaions: $A = C^*(M, \mathcal{F})$

$\mathcal{T}$: transverse bundle. Build $\nabla^{\mathcal{T}}$ an appropriate transverse connexion :

$$\nabla^{\mathcal{T}}\ :\ C_c^\infty(M) \to C_c^\infty(M, \mathcal{T}^*)\,.$$

Extend it on the holonomy graph $\mathcal{G}$ of the foliation

$$\nabla^{\tau}\ :\ C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}, \mathcal{T}^*)\,.$$

Proposition: $\nabla^{\mathcal{T}}$ can be chosen s.t. it acts as a derivation from $C_c^\infty(\mathcal{G})$ into $C_c^\infty(\mathcal{G}, \mathcal{T}^*)$.

C*-algebra: $C^*(M, \mathcal{F})$ = completion of $C_c^\infty(\mathcal{G})$ acting in $L^2(\mathcal{G})$.

$\to$ derivation $\nabla^{\tau}\ :\ C_c^\infty(\mathcal{G}) \to L^2(C_c^\infty(\mathcal{G}, \mathcal{T}^*))$

$\to$ Dirichlet form, transverse Laplacian, semigroup.

Bochner Laplacian on the Clifford C*-algebra (Davies-Rothaus) :

M Riemannian manifold, TM tangent bundle, $Cliff(TM)$ bundle of Clifford algebras of the tangent bundle:

$$Cliff(TM)_x = Cliff(T_xM)\,,\ x \in M\,.$$

Clifford algebra of E, f.d. euclidean real vector space $=$ algebra generated by 1 and E as a vector subspace + relations $\xi.\eta + \eta.\xi + 2(\eta|\xi)_E\,1 = 0$, $\xi, \eta \in E$.

Levi-Civita connexion

$\nabla : C_c^\infty(M, Cliff(TM)) \to C_c^\infty(M, Cliff(TM) \otimes T^*M)$

Fact : ∇ acts as a derivation.

Take

C*-algebra : $C_0(M, Cliff(TM)) = C^*_{Cliff}(M)$;

Bimodule: $L^2(Cliff(TM) \otimes T^*M)$;

Derivation: $\nabla =$ Levi-Civita connexion

$\to$ Dirichlet form

$\to$ Bochner heat semigroup $\varphi_t = e^{-t\nabla^*\nabla}$ on the Clifford C*-algebra of M.

Dirac heat semigroup on the Clifford C*-algebra :

Same context.

Dirac operator :

$D : C_c^\infty(M, Cliff(TM)) \to C_c^\infty(M, Cliff(TM))$

$$D(\xi) = \sum_i e_i.\nabla_{e_i}\xi$$

∇=Levi-Civita connexion ;
$\{e_1, .., e_m\}$ = any field of orthonormal basis of TM.

Fact : D is **not** a derivation.

Theorem. [CS]
1/ $\{e^{-tD^2}\}$ extends as a semigroup of completely bounded endomorphisms of $C^*_{Cliff}(M)$.
2/ It extends as a semigroup of completely positive contractions of $C^*_{Cliff}(M)$, if and only if the curvature operator is nonnegative.

Curvature tensor R :

$$R(v_1, v_2)v = -\big(\nabla_{v_1}\nabla_{v_2}v - \nabla_{v_2}\nabla_{v_1}v - \nabla_{[v_1,v_2]}v\big)$$

Curvature operator $\widehat{R}$:

$$\big(R(v_1, v_2)v_3\big|v_4\big)_{T_xM} = \big(\widehat{R}(v_1 \wedge v_2)\big|v_3 \wedge v_4\big)_{\Lambda^2_x(TM)}$$

$$\widehat{R} \in C^\infty(M, \Lambda^2(TM)) \text{ field of s.a. operators.}$$

N.B. $\qquad D^2 \longleftrightarrow \Delta_{Hodge} = d^*d + dd^*$

Harmonic structures on fractals.

Classical Dirichlet forms constructed by J. Kigami on "Post-critically finite self-similar sets".

Interest:

$\sharp\{$ eigenvalues of $\Delta \leq x \} = C.x^{d_S/2} + o(x^{d_S/2})\,,\ x \to +\infty$

(Kigami-Lapidus-Fukushima) with $1 < d_s < 2$

to be compared with Weyl expansion formula for a d-dimensional Riemannian manifold

$\sharp\{$ eigenvalues of $\Delta \leq x \} = c_d.vol(M)x^{d/2} + o(x^{d/2})$

$\rightarrow$ a lot of noncommutative geometry

Fredholm modules, integrability, conformal geometry with help of the Dixmier trace, and so on.

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