1. CHANGE OF BASIS FORMULAS.

1.1. **Notation.** Old basis $\mathfrak{B}$, new basis $\mathfrak{B}'$. Usually, the old basis is $\mathfrak{B} = (e_1, \ldots, e_n)$ (i.e., it is the "standard basis"), and the new basis is $\mathfrak{B}' = (v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are some n linearly independent vectors.

Recall that $[w]_{\mathfrak{B}'} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ means that $w = a_1 v_1 + \cdots + a_n v_n$. Also recall that

$$_{\mathfrak{B}'}[T]_{\mathfrak{B}'} = [[Tv_1]_{\mathfrak{B}'}, \ldots, [Tv_n]_{\mathfrak{B}'}]$$

is the matrix whose columns are the coordinates in the basis $\mathfrak{B}'$ of T applied to the elements of $\mathfrak{B}'$.

1.2. **Goal.** We assume that we know how to compute coordinates with respect to the old basis (this is very easy indeed if $\mathfrak{B}$ is the standard basis: in this case if $w = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then $[w]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, and if T is the linear transformation $T(v) = Av$, where A is some $n \times n$ matrix, then $_{\mathfrak{B}}[T]_{\mathfrak{B}} = A$). *The goal is to compute $[w]_{\mathfrak{B}'}$and $_{\mathfrak{B}'}[T]_{\mathfrak{B}'}$ in terms of $[w]_{\mathfrak{B}}$ and $_{\mathfrak{B}}[T]_{\mathfrak{B}}$.*

1.3. **Change of basis matrix.**

$$_{\mathfrak{B}}S_{\mathfrak{B}'} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = [[v_1]_{\mathfrak{B}}, \ldots, [v_n]_{\mathfrak{B}}].$$

Thus columns of $_{\mathfrak{B}}S_{\mathfrak{B}'}$ are elements of the new basis expressed in terms of the old basis.

Fact. $(_{\mathfrak{B}}S_{\mathfrak{B}'})^{-1} = {}_{\mathfrak{B}'}S_{\mathfrak{B}}$, *i.e., it is the matrix whose columns are elements of the old basis $\mathfrak{B}$ expressed in terms of the new basis $\mathfrak{B}'$.*

From this, you get many nice formulas:

$$\begin{aligned} [w]_{\mathfrak{B}'} &= {}_{\mathfrak{B}'}S_{\mathfrak{B}}[w]_{\mathfrak{B}} \\ {}_{\mathfrak{B}'}[T]_{\mathfrak{B}'} &= {}_{\mathfrak{B}'}S_{\mathfrak{B}}\ {}_{\mathfrak{B}}[T]_{\mathfrak{B}}\ {}_{\mathfrak{B}}S_{\mathfrak{B}'} = {}_{\mathfrak{B}'}S_{\mathfrak{B}}\ {}_{\mathfrak{B}}[T]_{\mathfrak{B}}\ ({}_{\mathfrak{B}'}S_{\mathfrak{B}})^{-1}. \end{aligned}$$

Exercise. (do not turn in). (a) Check that $[Tw]_{\mathfrak{B}'} = {}_{\mathfrak{B}'}[T]_{\mathfrak{B}'}\ [w]_{\mathfrak{B}'}$ (use the fact that $[Tw]_{\mathfrak{B}} = {}_{\mathfrak{B}}[T]_{\mathfrak{B}}\ [w]_{\mathfrak{B}}$).
(b) Make sense of $_{\mathfrak{B}}[T]_{\mathfrak{B}'}$ and $_{\mathfrak{B}'}[T]_{\mathfrak{B}}$ along the lines of the definition of $_{\mathfrak{B}'}[T]_{\mathfrak{B}'}$ (the first of these has as columns the coordinates in $\mathfrak{B}$ of the vectors obtained by applying T to the elements of $\mathfrak{B}'$). Find formulas for these in terms of $_{\mathfrak{B}}T_{\mathfrak{B}}$ and $_{\mathfrak{B}'}S_{\mathfrak{B}}$.
(c) What are $_{\mathfrak{B}}[I]_{\mathfrak{B}'}$ and $_{\mathfrak{B}'}[I]_{\mathfrak{B}}$, where I denotes the identity transformation $Iv = v$?