MATH 33A/2 PRACTICE MIDTERM 2

Problem 1. (True/False) ...

Problem 2. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
(a) Show that $\mathfrak{B} = (v_1, v_2, v_3)$ is a basis for $\mathbb{R}^3$.
(b) Let $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Find the coordinates of u with respect to the basis $\mathfrak{B}$.
(c) Find the change of basis matrix from the standard basis to $\mathfrak{B}$.
(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about the plane $x + y + z = 0$. Find the matrix of T with respec to $\mathfrak{B}$.
(e) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that T is invertible if and only if $\mathfrak{B}' = (u_1, u_2, u_3)$, where $u_1 = Tv_1$, $u_2 = Tv_2$ and $u_3 = Tv_3$, is a basis for $\mathbb{R}^3$.
(f) With the same notation as in (e), assume that T is invertible. Express the change of basis matrix from the standard basis to the basis $\mathfrak{B}'$.

Problem 3. (a) Prove that if V, W are subspaces of $\mathbb{R}^n$ and V is a subspace of W, then $W^\perp$ is a subspace of $V^\perp$. Prove that $\mathbb{R}^n = V^\perp + V$ and that $V \cap V^\perp = 0$. (b) Let $V \subset \mathbb{R}^5$ be a subspace, and let W consist of the zero vector and also all vectors w in $\mathbb{R}^5$ that make the angle 30 degrees with every vector in V. Is W a subspace? Prove your answer.

Problem 4. (a) Let V be the subspace of $\mathbb{R}^4$ consisting of all vectors $v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ for which $x + y + z = w$. Find an orthonormal basis for V. (b) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Explain why the Gramm-Schmidt procedure applied to (v_1, v_2, v_3) does not give you 3 orthonormal vectors.

Problem 5. (a) A matrix A is called symmetric if $A = A^T$. Find all 2×2 symmetric matrices which are also orthogonal. What about 3×3? (b) Give an example of a 3×3 matrix which is not orthogonal, but which is invertible and compute its QR factorization. (c) For which scalars c_1, c_2, c_3 is the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & c_1 & c_2 \\ 5 & 6 & c_3 \end{bmatrix}$$

orthogonal?

Problem 6. Find the least squares solution to the system of equations

$$\begin{aligned} x+y+z &= 1 \\ x+y+z &= 2 \\ x+y-z &= 3. \end{aligned}$$

Problem 7. A nonzero matrix A is called "nilpotent" if $A^k = 0$ for some integer $k > 0$. (a) Show that the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent. (b) Can a nilpotent matrix be orthogonal? (c) Show that if A is nilpotent, then the transpose A^t is also nilpotent (d) [Harder] Can a nilpotent matrix be symmetric? (Hint: try it first if $T \neq 0$ but $T^2 = 0$; try to use (c) and the formula relating the kernel of T and the range of the transpose).

There will be more problems on determinants added later; for now see h/w for chapter 6.