MIDTERM 1

May 20, 2002

Instructions.

Please show your work. You will receive little or no credit for an answer not accompanied by appropriate explanations, even if the answer is correct. If you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You will find a list of some useful formulas on page 2 of the exam.

At the end of the exam, please hand the exam paper to your TA. Please be prepared to show your university ID upon request.

If you have a question about the grading of a particular problem, please come and see me or one of the TAs *within 14 days of the exam*.

Name:__________________________ Student ID:_____________

Section:______

#1	#2	#3	#4	#5	Total

$$\cos^2 t + \sin^2 t = 1 \qquad \sin 2t = 2 \sin t \cos t$$

$$\sin^2 \tfrac{t}{2} = \tfrac{1-\cos t}{2} \qquad \cos^2 \tfrac{t}{2} = \tfrac{1+\cos t}{2}$$

$$\cos 2t = \cos^2 t - \sin^2 t \qquad \tfrac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \tfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$L = \int_a^b \|\overrightarrow{r}\| dt \qquad \tfrac{d}{dt}\langle f, g, h\rangle = \langle f', g', h'\rangle$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta \qquad \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\theta|$$

Problem 1. Let $f(x,y) = \frac{x}{y}$. (a) Determine the domain and range of f. (b) Sketch enough level lines of f to give an idea of how the level lines look like. (c) Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution. (a) Since the formula defining f makes sense when $y \neq 0$, the domain of f is all points (x,y) for which $y \neq 0$. In other words, these are all points of the plane not lying on the x-axis. (b) The level line correspoding to the value c is the set of solutions to the equation

$$f(x,y) = c.$$

Thus we solve

$$\frac{x}{y} = c$$

to get

$$x = cy.$$

Remembering that $y \neq 0$, we find that the level lines are the pairs of rays eminating from the origin (but not containing the origin) with slopes $1/c$ for all $c \in (-\infty, \infty)$.

(c) Set $x = t$, $y = t$. Then

$$\lim_{t\to 0} f(x,y) = \lim_{t\to 0} \frac{t}{t} = 1.$$

Set instead $x = 0$, $y = t$. Then

$$\lim_{t\to 0} f(x,y) = \lim_{t\to 0} \frac{0}{t} = 0.$$

So the limit does not exist.

Problem 2. Evaluate the following limit, or show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}.$$

Solution. Since

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

we know that given $\varepsilon > 0$ there is a $\delta > 0$ so that if $0 < |t| < \delta$, we have that

$$\left| \frac{\sin t}{t} - 1 \right| < \varepsilon.$$

Thus if $0 < |x+y| < \delta$, we know that

$$\left| \frac{\sin(x+y)}{(x+y)} - 1 \right| < \varepsilon.$$

The conditions of the definition of the limit are satisfied, so we know that the limit exists and equals 1.

Problem 3. A ball is thrown from the ground at the angle of $\pi/6$ with the ground at the initial velocity of $10m/s$. The acceleration of gravity is $g = 9.8m/s^2$. Find how long it takes before the ball hits the ground again.

Solution. Choose a coordinate system for which the ball is initially at the origin, the x-axis is parallel to the ground, and the y-axis is perpendicular to the ground, pointing up. Let $\overrightarrow{r}(t)$ denote the position of the ball at time t. The acceleration of the ball is then $\overrightarrow{r}''(t)$. Since we are in a free fall, the acceleration is equal to the acceleration of gravity, which is a vector pointing downward and of magnitude g. So we know that $\overrightarrow{r}''(t) = g$. Integrating we get

$$\begin{aligned}\overrightarrow{r}'(t) &= \overrightarrow{v_0} - gt\overrightarrow{j} \\ \overrightarrow{r}(t) &= \overrightarrow{r_0} + \overrightarrow{v_0}t - \frac{1}{2}gt^2\overrightarrow{j}.\end{aligned}$$

The initial conditions imply that $\overrightarrow{r_0} = 0$ and that $\overrightarrow{v_0}$ is the initial velocity. Thus we know that the magnitude of $\overrightarrow{v_0}$ is 10 and that it makes the angle of $\pi/6$ with the x-axis. Thus $\overrightarrow{v_0} = \langle 10\cos\pi/6, 10\sin\pi/6\rangle = \langle 5\sqrt{3}, 5\rangle$. It follows that

$$\overrightarrow{r}(t) = \langle 5\sqrt{3}\cdot t, 5t - \frac{1}{2}gt^2\rangle.$$

The ball hits the ground again when $5t - \frac{1}{2}gt^2 = 0$, i.e., $5 = \frac{1}{2}gt$, so that

$$t = 10/g \cong 1.02s.$$

Problem 4. Sketch the curves obtained as intersections of the quadratic surface $x^2 - 3y^2 + z = 0$ with the planes $z =$ const. Name and sketch the quadratic surface.

Solution. If $z = c$ we get the equation

$$x^2 - 3y^2 = -c.$$

For $c > 0$ this is a hyperbola, oriented along the y-axis. For $c < 0$this is a hyperbola, oriented along the x-axis. The surface is a hyperbolic paraboloid. It is sketched on p.858 of the book.

Problem 5. Let $f(x,y) = e^{-(x^2+y^2)}$. Compute the following partial derivatives of f:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}.$$

Solution. We have:

$$\frac{\partial f}{\partial x} = -2xe^{-(x^2+y^2)}, \quad \frac{\partial f}{\partial y} = -2ye^{-(x^2+y^2)}.$$

Also,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(-2xe^{-(x^2+y^2)}) = -2(-2y)e^{-(x^2+y^2)} = 4xye^{-(x^2+y^2)}.$$