PRACTICE MIDTERM 1

PLEASE NOTE:

The midterm will be in room 2160E Dickson, NOT the usual classroom.

Instructions.

Please show your work. You will receive little or no credit for an answer not accompanied by appropriate explanations, even if the answer is correct. If you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You will find a list of some useful formulas on page 2 of the exam.

At the end of the exam, please hand the exam paper to your TA. Please be prepared to show your university ID upon request.

If you have a question about the grading of a particular problem, please come and see me or one of the TAs *within 14 days of the exam*.

#1	#2	#3	#4	#5	Total

$$\cos^2 t + \sin^2 t = 1 \qquad \sin 2t = 2\sin t\cos t$$

$$\sin^2 \tfrac{t}{2} = \tfrac{1-\cos t}{2} \qquad \cos^2 \tfrac{t}{2} = \tfrac{1+\cos t}{2}$$

$$\cos 2t = \cos^2 t - \sin^2 t \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$L = \int_a^b \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} dt \qquad A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 d\theta$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\|\cos\theta \qquad \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\theta|$$

Problem 1. Find an equation of the plane through the points $(1,1,1)$, $(0,1,1)$ and $(-1,-1,-1)$.

Solution: Denote the points by A, B and C, respectively. The vectors $\overrightarrow{BA} = \langle 1,0,0\rangle$ and $\overrightarrow{CA} = \langle 2,2,2\rangle$ lie in the plane. Thus a normal vector to the plane can be found as

$$\overrightarrow{n} = \langle 1,0,0\rangle \times \langle 2,2,2\rangle = 2(\overrightarrow{i} \times (\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k})) = 2(\overrightarrow{k} - \overrightarrow{j}) = \langle 0,-2,2\rangle.$$

Since the point $(-1,-1,-1)$ lies in the plane, we find that the equation is

$$0 \cdot (x-(-1)) + (-2)(y-(-1)) + 2(z-(-1)) = 0,$$

or

$$-2(y+1) + 2(z+1) = 0.$$

Problem 2. (a) Prove that the vectors $\langle 1,4,7\rangle$, $\langle 2,5,8\rangle$ and $\langle 3,6,9\rangle$ are parallel to the same plane. (b) Find an equation of such a plane. (c) Which room will the midterm exam be in? (Hint: see the class web page!)

Solution: (a) Consider the parallelepiped determined by these three vectors. Its volume is zero exactly if the vectors are all parallel to the same plane. The volume is given by the determinant

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0.$$

(b) A vector perpendicular to this plane will be perpendicular to all three of the given vectors. So we could take as the normal vector the cross product of the first two:

$$\begin{aligned} \langle 1,4,7\rangle \times \langle 2,5,8\rangle &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{vmatrix} \\ &= \overrightarrow{i}\cdot(4\cdot 8-5\cdot 7)-\overrightarrow{j}\cdot(1\cdot 8-2\cdot 7)+\overrightarrow{k}\cdot(1\cdot 5-4\cdot 2) \\ &= \langle -3,-6,-3\rangle = -3\cdot\langle 1,2,1\rangle. \end{aligned}$$

Thus the vector $\langle 1,2,1\rangle$ is perpendicular to the plane in question. Thus we could take, e.g., the plane

$$x+2y+z=0.$$

(c) 2160E Dickson.

Problem 3. Let Cbe the parametric curve $x = \sin t$, $y = 3\cos t$. Sketch this curve. Find all points on the curve at which the tangent line to the curve is parallel to the line $y = x$. Where is Dickson, anyway? (Hint: it's near the sculpture garden).

Solution: The curve looks like an ellipse; indeed, it is given by the Cartesian equation $x^2 + (y/3)^2 = 1$.
The slope of the curve at time t is given by

$$s = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{\cos t} = -3\tan t.$$

For the tangent line to be parallel to the line

$$y = x,$$

the curve must have slope 1 at that point. For the slope to be 1 we must have

$$\tan t = \frac{-1}{3},$$

so that $t = \tan^{-1}(-1/3)$.

Problem 4. Find an equation in polar coordinates that describes the line $y = 3x+1$.

Solution: Let's say that $r = f(\theta)$. Then we will have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Thus

$$f(\theta)\sin\theta = y = 3x+1 = 3f(\theta)\cos\theta+1.$$

Solving this for $f(\theta)$gives

$$f(\theta)(\sin\theta - 3\cos\theta) = 1$$

so that

$$f(\theta) = \frac{1}{\sin\theta - 3\cos\theta}.$$

Problem 5. For which values of c is the following curve an ellipse, and for which values is it a hyperbola: $x^2+cy^2+2\sqrt{|c|}y-1=0$? Sketch the curve for $c=1,-1,0$.

Solution: For $c>0$ we get

$$x^2+cy^2+2\sqrt{c}+1-2=x^2+(\sqrt{c}y+1)^2-2$$

so that the equation becomes

$$x^2+c(y+c^{-\frac{1}{2}})^2=2.$$

This is the equation of an ellipse (shifted down by $-c^{-\frac{1}{2}}$ along the y-axis). If $c=1$, the ellipse becomes

$$\frac{x^2}{(\sqrt{2})^2}+\frac{(y+1)^2}{(\sqrt{2})^2}=1.$$

This is a circle of radius $\sqrt{2}$centered at $(0,-1)$.

For $c=0$ the equation becomes $x^2=1$; the resulting curve consists of two lines, $x=\pm 1$ (with $y$arbitrary).

For $c<0$the equation becomes $x^2-\sqrt{|c|}(y-1)^2=0$. The curve is a pair of lines, $x=\pm\sqrt{|c|}(y-1)$. For $c=-1$, the lines are $x=y-1$ and $x=-y+1$.

Thus the curve is an ellipse for $c>0$ and is never a hyperbola.