MIDTERM 1

April 24, 2002

Instructions.

Please show your work. You will receive little or no credit for an answer not accompanied by appropriate explanations, even if the answer is correct. If you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You will find a list of some useful formulas on page 2 of the exam.

At the end of the exam, please hand the exam paper to your TA. Please be prepared to show your university ID upon request.

If you have a question about the grading of a particular problem, please come and see me or one of the TAs *within 14 days of the exam*.

Name:__________________________ Student ID:_____________

#1	#2	#3	#4	#5	Total

$$\cos^2 t + \sin^2 t = 1 \qquad \sin 2t = 2\sin t \cos t$$
$$\sin^2 \tfrac{t}{2} = \tfrac{1-\cos t}{2} \qquad \cos^2 \tfrac{t}{2} = \tfrac{1+\cos t}{2}$$
$$\cos 2t = \cos^2 t - \sin^2 t \qquad \tfrac{dy}{dx} = \tfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$L = \int_a^b \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} dt \qquad A = \int_a^b \tfrac{1}{2}[f(\theta)]^2 d\theta$$
$$\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta \qquad \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\theta|$$

Problem 1. Let $\overrightarrow{v}$ be the vector $\langle 1,2,3\rangle$ and $\overrightarrow{w}$ be the vector $\langle 1,1,0\rangle$.
(a) Find the unit vector in the same direction as $\overrightarrow{v}$.
(b) Find the angle between the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$.
(c) Let ℓ_1 be the line through $(0,0,0)$ in the direction of $\overrightarrow{v}$ and ℓ_2 be the line through $(-1,0,3)$ in the direction of $\overrightarrow{w}$. Write down the parametric equations of ℓ_1 and ℓ_2. Then determine if these lines are parallel, skew, or intersecting.

Solution. (a) The length of $\overrightarrow{v}$ is $\sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$. Thus the desired unit vector is $\frac{\overrightarrow{v}}{\sqrt{14}}$.

(b) We have that

$$\cos\theta = \frac{\overrightarrow{v}\cdot\overrightarrow{w}}{\|\overrightarrow{v}\|\|\overrightarrow{w}\|} = \frac{1\cdot 1+2\cdot 1+3\cdot 0}{\sqrt{14}\sqrt{1^2+1^2+0^2}} = \frac{3}{\sqrt{28}} = \frac{3}{2\sqrt{7}}.$$

So $\theta = \cos^{-1}\frac{3}{2\sqrt{7}}$.

(c) ℓ_1: $x=t$, $y=2t$, $z=3t$, $-\infty < t < +\infty$; ℓ_2: $x=s-1$, $y=s$, $z=3$, $-\infty < s < \infty$. Since the angle between the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is not 0, the vectors are not parallel, and so the lines ℓ_1 and ℓ_2 are not parallel. Let us check if they are intersecting. At the point of intersection we must have

$$\begin{aligned} x &= t = s-1 \\ y &= 2t = s \\ z &= 3t = 3. \end{aligned}$$

From the last equation $t=1$; thus $s=2t=2$. The first equation is satisfied, so that the lines do intersect at $(1,2,3)$.

Problem 2. Let $\overrightarrow{v}$ be the vector $\langle 1, 1, -1\rangle$ and $\overrightarrow{w}$ be the vector $\langle 1, 1, 1\rangle$. Find the equation of the plane parallel to both $\overrightarrow{v}$ and $\overrightarrow{w}$, and passing through the point $(-1, -1, -1)$.

Solution. We compute the cross product $\overrightarrow{v} \times \overrightarrow{w} = \langle 2, -2, 0\rangle$. This gives us a normal vector to the plane. Thus the equation is

$$2(x + 1) + 2(y + 1) + 0(z + 1) = 0,$$

or

$$2x - 2y = 0.$$

Problem 3. Find an equation in polar coordinates that describes the line $y = 3x + 1$. In your answer, please indicate the range of the parameter θ.

Solution. Let's say that $r = f(\theta)$. Then we will have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Thus

$$f(\theta)\sin\theta = y = 3x + 1 = 3f(\theta)\cos\theta + 1.$$

Solving this for $f(\theta)$gives

$$f(\theta)(\sin\theta - 3\cos\theta) = 1$$

so that

$$f(\theta) = \frac{1}{\sin\theta - 3\cos\theta}.$$

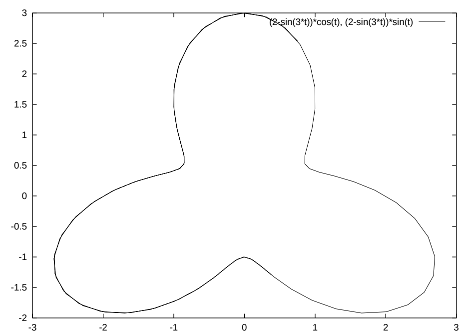
For $f(\theta)$ to make sense, we must have $\sin\theta \neq 3\cos\theta$, i.e., $\theta \neq \tan^{-1} 3$. Thus the range of $\theta$is from 0 to 2π, excluding all values of $\tan^{-1} 3$.

Problem 4. Consider the polar curve

$$r = 2 - \sin 3\theta.$$

(a) Sketch the curve.
(b) Find the area between the x-axis and the part of the curve which lies below the x axis.

Solution.(a) The graph is shown below:



(b) From the picture, we see that the part of the curve that lies below the x axis corresponds to $\pi \leq \theta \leq 2\pi$. Thus the area is

$$\begin{aligned}
A &= \int_{\pi}^{2\pi} \frac{1}{2}(2 - \sin 3\theta)^2 d\theta \\
&= \frac{1}{2}\int_{\pi}^{2\pi} (4 - 4\sin 3\theta + \sin^2 3\theta) d\theta \\
&= \frac{1}{2} \cdot 4\pi + 2 \cdot \frac{\cos 3\theta}{3}\Big|_{\pi}^{2\pi} + \frac{1}{2}\int_{\pi}^{2\pi} \frac{1 - \cos 6\theta}{2} d\theta \\
&= \frac{9\pi}{4} + \frac{4}{3}.
\end{aligned}$$

Problem 5. Sketch the following planar curve, indicating foci, intersections with the coordinate axes and asymptotes, as appropite: $x^2 - 25y^2 + 5 = 0$.

Solution.Rewriting the equation we get

$$\frac{y^2}{(1/\sqrt{5})^2} - \frac{x^2}{(\sqrt{5})^2} = 1.$$

From this we find that the foci are located on the y-axis, at $y = \pm\sqrt{\frac{1}{5} + 5} = \pm\sqrt{\frac{26}{5}}$. The hyperbola intersects the y axis at $y = \pm\sqrt{\frac{1}{5}}$. Asymptotically, $y = \pm\frac{1}{5}x$. The hyperbola looks like this: