PRACTICE FINAL EXAM

June 13, 2002

Instructions.

Please show your work. You will receive little or no credit for an answer not accompanied by appropriate explanations, even if the answer is correct. If you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You will find a list of some useful formulas on page 2 of the exam.

At the end of the exam, please hand the exam paper to your TA. Please be prepared to show your university ID upon request.

Name:__________________________ Student ID:_____________

Section:______

#1	#2	#3	#4	#5	#6	#7
#8	**#9**	**#10**	**#11**	**#12**	**#13**	**Total**

$$\cos^2 t + \sin^2 t = 1 \qquad \sin 2t = 2\sin t \cos t$$

$$\sin^2 \frac{t}{2} = \frac{1-\cos t}{2} \qquad \cos^2 \frac{t}{2} = \frac{1+\cos t}{2}$$

$$\cos 2t = \cos^2 t - \sin^2 t \qquad \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$L = \int_a^b \|\overrightarrow{r}'\| dt \qquad \frac{d}{dt}\langle f, g, h\rangle = \langle f', g', h'\rangle$$

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\|\|\overrightarrow{w}\| \cos\theta \qquad \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\theta|$$

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|} \qquad \kappa = \frac{\|\overrightarrow{T}'(t)\|}{\|\overrightarrow{r}'(t)\|}$$

Problem 1. Find an equation of the plane through the points $(1,1,1)$, $(0,1,1)$ and $(-1,-1,-1)$.

Solution: Denote the points by A, B and C, respectively. The vectors $\overrightarrow{BA} = \langle 1,0,0\rangle$ and $\overrightarrow{CA} = \langle 2,2,2\rangle$ lie in the plane. Thus a normal vector to the plane can be found as

$$\overrightarrow{n} = \langle 1,0,0\rangle \times \langle 2,2,2\rangle = 2(\overrightarrow{i} \times (\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k})) = 2(\overrightarrow{k} - \overrightarrow{j}) = \langle 0,-2,2\rangle.$$

Since the point $(-1,-1,-1)$ lies in the plane, we find that the equation is

$$0 \cdot (x-(-1)) + (-2)(y-(-1)) + 2(z-(-1)) = 0,$$

or

$$-2(y+1) + 2(z+1) = 0.$$

Problem 2. (a) Prove that the vectors $\langle 1,4,7\rangle$, $\langle 2,5,8\rangle$ and $\langle 3,6,9\rangle$ are parallel to the same plane. (b) Find an equation of such a plane.

Solution: (a) Consider the parallelepiped determined by these three vectors. Its volume is zero exactly if the vectors are all parallel to the same plane. The volume is given by the determinant

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0.$$

(b) A vector perpendicular to this plane will be perpendicular to all three of the given vectors. So we could take as the normal vector the cross product of the first two:

$$\begin{aligned} \langle 1,4,7\rangle \times \langle 2,5,8\rangle &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{vmatrix} \\ &= \overrightarrow{i}\cdot(4\cdot 8 - 5\cdot 7) - \overrightarrow{j}\cdot(1\cdot 8 - 2\cdot 7) + \overrightarrow{k}\cdot(1\cdot 5 - 4\cdot 2) \\ &= \langle -3,-6,-3\rangle = -3\cdot\langle 1,2,1\rangle. \end{aligned}$$

Thus the vector $\langle 1,2,1\rangle$ is perpendicular to the plane in question. Thus we could take, e.g., the plane

$$x + 2y + z = 0.$$

Problem 3. Let Cbe the parametric curve $x = \sin t$, $y = 3\cos t$. Sketch this curve. Find all points on the curve at which the tangent line to the curve is parallel to the line $y = x$.

Solution: The curve looks like an ellipse; indeed, it is given by the Cartesian equation $x^2 + (y/3)^2 = 1$.
The slope of the curve at time t is given by

$$s = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{\cos t} = -3\tan t.$$

For the tangent line to be parallel to the line

$$y = x,$$

the curve must have slope 1 at that point. For the slope to be 1 we must have

$$\tan t = \frac{-1}{3},$$

so that $t = \tan^{-1}(-1/3)$.

Problem 4. Find an equation in polar coordinates that describes the line $y = 3x + 1$.

Solution: Let's say that $r = f(\theta)$. Then we will have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Thus

$$f(\theta)\sin\theta = y = 3x + 1 = 3f(\theta)\cos\theta + 1.$$

Solving this for $f(\theta)$gives

$$f(\theta)(\sin\theta - 3\cos\theta) = 1$$

so that

$$f(\theta) = \frac{1}{\sin\theta - 3\cos\theta}.$$

Problem 5. Let $f(x, y) = \frac{x}{y}$. (a) Determine the domain and range of f. (b) Sketch enough level lines of f to give an idea of how the level lines look like. (c) Does the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution. (a) Since the formula defining f makes sense when $y \neq 0$, the domain of f is all points (x, y) for which $y \neq 0$. In other words, these are all points of the plane not lying on the x-axis. (b) The level line correspoding to the value c is the set of solutions to the equation

$$f(x, y) = c.$$

Thus we solve

$$\frac{x}{y} = c$$

to get

$$x = cy.$$

Remembering that $y \neq 0$, we find that the level lines are the pairs of rays eminating from the origin (but not containing the origin) with slopes $1/c$ for all $c \in (-\infty, \infty)$.

(c) Set $x = t$, $y = t$. Then

$$\lim_{t\to 0} f(x, y) = \lim_{t\to 0} \frac{t}{t} = 1.$$

Set instead $x = 0$, $y = t$. Then

$$\lim_{t\to 0} f(x, y) = \lim_{t\to 0} \frac{0}{t} = 0.$$

So the limit does not exist.

Problem 6. Evaluate the following limit, or show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}.$$

Solution. Since

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

we know that given $\varepsilon > 0$ there is a $\delta > 0$ so that if $0 < |t| < \delta$, we have that

$$\left|\frac{\sin t}{t} - 1\right| < \varepsilon.$$

Thus if $0 < |x+y| < \delta$, we know that

$$\left|\frac{\sin(x+y)}{(x+y)} - 1\right| < \varepsilon.$$

The conditions of the definition of the limit are satisfied, so we know that the limit exists and equals 1.

Problem 7. Let $f(x, y) = e^{-(x^2+y^2)}$. Compute the following partial derivatives of f:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}.$$

Solution. We have:

$$\frac{\partial f}{\partial x} = -2xe^{-(x^2+y^2)}, \quad \frac{\partial f}{\partial y} = -2ye^{-(x^2+y^2)}.$$

Also,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(-2xe^{-(x^2+y^2)}) = -2(-2y)e^{-(x^2+y^2)} = 4xye^{-(x^2+y^2)}.$$

Problem 8. Find the maximum value of xyz if $x+y+z = 1$ and $x, y, z \geq 0$.

Solution: It is enough to maximize $f(x, y) = xy(1 - x - y)$ over all x, y so that $x, y \geq 0$ and $1 - x - y \leq 0$. The region described by these inequalities is a triangle, so we must maximize f inside the triangle and also on its boundary. However, on the boundary, one of the following three equations hold: either $x = 0$, or $y = 0$, or $1 - x - y = 0$ (so that $z = 0$). Thus f is identically zero on the boundary. Since zero is clearly not the maximal value of f (e.g., $f(1/2, 1/3, 1/6) > 0$), it is enough to consider f inside of the region only.

We compute the partial derivatives of f and set them equal to zero:

$$\begin{aligned} y(1 - x - y) - xy &= 0 \\ x(1 - x - y) - xy &= 0 \end{aligned}$$

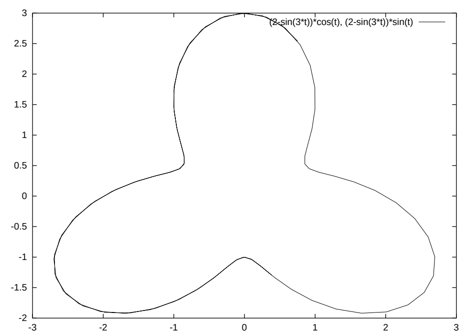
From the first equation we get that either $y = 0$, or $x = (1 - x - y)$. As we noted before, $y = 0$ cannot be a point of maximum. So $x = 1 - x - y$. Similarly, we get from the second equation that $1 - x - y = y$ (since y cannot be zero). We conclude that $x = 1 - x - y = y$. Thus $x = y$ and $x = 1 - x - x = 1 - 2x$. Thus $3x = 1$ and $x = 1/3 = y$. So the maximum value is $f(1/3, 1/3) = 1/27$.

Problem 9. Consider the polar curve

$$r = 2 - \sin 3\theta.$$

(a) Sketch the curve.
(b) Find the area between the x-axis and the part of the curve which lies below the x axis.

Solution.(a) The graph is shown below:



(b) From the picture, we see that the part of the curve that lies below the x axis corresponds to $\pi \leq \theta \leq 2\pi$. Thus the area is

$$\begin{aligned} A &= \int_{\pi}^{2\pi} \frac{1}{2}(2 - \sin 3\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\pi}^{2\pi} (4 - 4\sin 3\theta + \sin^2 3\theta) d\theta \\ &= \frac{1}{2} \cdot 4\pi + 2 \cdot \frac{\cos 3\theta}{3}\Big|_{\pi}^{2\pi} + \frac{1}{2}\int_{\pi}^{2\pi} \frac{1 - \cos 6\theta}{2} d\theta \\ &= \frac{9\pi}{4} + \frac{4}{3}. \end{aligned}$$

Problem 10. Find all points on the surface $x^2 + 3y + z^2 = 1$ at which the tangent plane is parallel to the xz-plane.

Solution: The normal vector to the surface $f(x, y, z) = \text{const}$ for $f(x, y, z) = x^2 + 3y + z^2$ is given by

$$\nabla f = \langle 2x, 3, 2z \rangle.$$

We want the tangent plane to the surface to be parallel to the xz-plane; that is to say, we want the normal vector to be parallel to the y plane. Thus its x and z components must be zero. Thus we get $2x = 0$ and $2z = 0$, so that $x = z = 0$. From the equation for the surface we get that $3y = 1$, so that $y = 1/3$. Thus the only point with the required property is the point $(0, 1/3, 0)$.

Problem 11. Let $f(x, y) = \sin xy$. Find the minimum and maximum values of f in the region $x^2 + y^2 \leq 1$.

Solution: We first try to find the min/max values of f inside the disk $x^2 + y^2 \leq 1$. We compute

$$\nabla f = \langle y \cos xy, x \cos xy \rangle.$$

Thus

$$\nabla f = 0$$

means that either $\cos xy = 0$ (which means that $xy = \pm\frac{\pi}{2}$, and which is impossible, since $x^2 + y^2 \leq 1$, so that $|x| \leq 1$, $|y| \leq 1$and thus $|xy| \leq 1 < \pi/2$), or $x = y = 0$. Thus the only critical point inside of the region of interest is $x = y = 0$. The value of the function at that critical point is 0.

Next, we check for the min/max on the boundary. Our constrait is that $F(x, y) = 0$, where $F(x, y) = x^2 + y^2 - 1$. Using Lagrange multipliers, we get the following system of equations:

$$\begin{cases} \nabla f = \lambda \nabla F \\ F = 0 \end{cases}$$

which means that

$$\begin{aligned} y \cos xy &= \lambda 2x \\ x \cos xy &= \lambda 2y \\ x^2 + y^2 &= 1. \end{aligned}$$

If $x = 0$, we get that $\cos xy = 1$, so from the first equation $y = 0$; this is impossible since $x^2 + y^2 = 1$. Similarly, if $y = 0$ we get $x = 0$, which is again impossible. So both x and y are nonzero. Thus

$$2\lambda \frac{x}{y} = \cos xy = 2\lambda \frac{y}{x}.$$

If $\lambda = 0$, we get that $\cos xy = 0$(since $x, y \neq 0$). This is impossible as we saw before. Thus $\lambda \neq 0$ and we get that

$$\frac{x}{y} = \frac{y}{x}.$$

Thus $x^2 = y^2$; so from the last equation we get $x^2 + y^2 = 2x^2 = 1$, so $x^2 = y^2 = \frac{1}{2}$.

Thus the possible locations of max/min are the four points

$$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

Evaluating f at these gives $\sin \frac{1}{2}$ for the maximum value and $-\sin \frac{1}{2}$ for the minimum value.

Problem 12. Find all critical points of

$$f(x, y) = x^2 + y^2 + y^3 + 3xy^2$$

and classify them as local minima, local maxima, or saddle points.

Solution: We set $\nabla f = 0$, to get

$$\begin{aligned} 2x + 3y^2 &= 0 \\ 3y^2 + 6xy + 2y &= 0 \end{aligned}$$

The second equation reads $3y(y + 2x + 2) = 0$, so either $y = 0$, or $y = -2x - 2$. If $y = 0$, the first equation gives $x = 0$. If $y = -2x - 2$, the first equation gives

$$2x + 12(x - 1)^2 = 0,$$

so that $x + 6x^2 - 12x + 6 = 6x^2 - 11x + 6 = 0$. The solutions are

$$x = \frac{11 \pm \sqrt{121 - 36 \cdot 4}}{12},$$

so there are no real solutions.

So the only critical point is $(0, 0)$.

The discriminant of f at $(0, 0)$is given by the matrix

$$\begin{vmatrix} 2 & 6y \\ 6y & 6y + 6x + 2 \end{vmatrix} = \begin{vmatrix} 2 & 6y \\ 6y & 6y + 6x + 2 \end{vmatrix} = 4.$$

Since the discriminant is positive and f_{xx} is positive, the point is a local minimum.

Problem 13. Find the points on the curve $\frac{1}{4}x^2 + y^2 = 1$ at which the curvature is maximal and find the maximal value of the curvature.

Solution: Consider a parameterization of the ellipse,

$$\begin{aligned} x &= 2\cos t \\ y &= \sin t \\ z &= 0, \end{aligned}$$

$0 \leq t \leq 2\pi$. The tangent vector to the curve is given by

$$\overrightarrow{r}'(t) = \langle -2\sin t, \cos t, 0 \rangle.$$

The second derivative is then

$$\overrightarrow{r}''(t) = \langle -2\cos t, -\sin t, 0 \rangle.$$

Thus

$$\overrightarrow{r'} \times \overrightarrow{r}''(t) = (2\sin^2 t + 2\cos^2 t)\overrightarrow{j} = 2\overrightarrow{j}.$$

Hence the curvature is

$$\kappa(t) = \frac{\|\overrightarrow{r'} \times \overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|^3} = \frac{2}{\sqrt{4\sin^2 t + \cos^2 t}} = \frac{2}{\sqrt{1 + 3\sin^2 t}}.$$

The curvature is maximal if $\sqrt{1 + 3\sin^2 t}$ is minimal. This minimum value occurs when $\sin t = 0$. Thus the curvature is maximal when $\sin t = 0$, i.e., $t = 0$ or $t = \pi$. The corresponding points on the ellipse are $(2, 0)$ and $(0, 2)$; the corresponding value of the curvature is 2.