# MATH 32A FINAL EXAMINATION

June 13, 2002

**Instructions.**

Please show your work. You will receive little or no credit for an answer not accompanied by appropriate explanations, even if the answer is correct. If you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You will find a list of some useful formulas on page 2 of the exam.

At the end of the exam, please hand the exam paper to your TA. Please be prepared to show your university ID upon request.

Name: Sample Student ID: 999999999

Section:None

| #1 | #2 | #3 | #4 | #5 | #6 | #7 |
|---|---|---|---|---|---|---|
| | | | | | | |
| #8 | #9 | #10 | #11 | #12 | #13 | Total |
| | | | | | | |

$$\cos^2 t + \sin^2 t = 1 \qquad \sin 2t = 2\sin t\cos t$$

$$\sin^2\frac{t}{2} = \frac{1-\cos t}{2} \qquad \cos^2\frac{t}{2} = \frac{1+\cos t}{2}$$

$$\cos 2t = \cos^2 t - \sin^2 t \qquad \frac{\partial f}{\partial x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$L = \int_a^b \|\vec{r}\,'\|dt \qquad \frac{d}{dt}\langle f,g,h\rangle = \langle f',g',h'\rangle$$

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta \qquad \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\||\sin\theta|$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \qquad \kappa = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$$

**Problem 1.** Let

$$f(x,y) = \sin^2(x^2 + y^2).$$

Sketch level lines corresponding to $f(x,y) = -1, 0, 1$. Indicate maxima and minima of the function.

Solution: The level set corresponding to the value $-1$ is empty, since $f \geq 0$. The level set corresponding to the value of 0 consists of solutions to $0 = \sin^2(x^2 + y^2)$ and corresponds to $x^2 + y^2 = n\pi$. The lines are circles of radius $\sqrt{\pi n}$, $n = 1, 2, 3\ldots$ as well as the point at the origin. The level set corresponding to the value of 1 consists of solutions to $1 = \sin^2(x^2 + y^2)$ and corresponds to $x^2 + y^2 = \frac{\pi}{2} + 2n\pi$. The lines are circles of radius $\sqrt{\frac{\pi}{2} + 2\pi n}$, $n = 1, 2, \ldots$. The maximum of the function is 1 and the minimum is 0.

**Problem 2.** Evaluate the following limit, or show that the limit does not existx

$$\lim_{(x,y)\to 0} \frac{x\sqrt{|y|}}{x^2+y}.$$

Solution: Let $y = t^2$, $x = t$. The limit is

$$\lim_{t\to 0} \frac{t\sqrt{t^2}}{t^2+t^2} = \frac{t|t|}{2t^2} = \frac{|t|}{2t},$$

which does not exist. Thus the limit does not exist.

**Problem 3.** For each of the three curves below, find an equation $r = f(\theta)$ in polar coordinates representing the same curve, or explain why no such equation can exist.

(a) Circle of radius 2 centered at the origin. (b) The line $x = y + 1$. (c) The line $y = x$.

(a) The corresponding equation is $r = 2$. (b) Using the equations $x = r\cos\theta$, $y = r\sin\theta$, we get

$$r\cos\theta = r\sin\theta + 1$$

so that

$$r(\cos\theta - \sin\theta) = 1$$

so that

$$r = \frac{1}{\cos\theta - \sin\theta}.$$

(c) There is no equation of the kind the problem asks for. All points on the curve $y = x$ correspond to $\theta = \pi/4 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Thus if this curve were to be given by the equation $r = f(\theta)$, the set of values of $\theta$ where $f(\theta)$is non-zero would have to be contained in the set $\pi/4 + n\pi$. Thus $|f(\theta)|$ would have to assume all values in $(0, \infty)$ as $\theta$ ranges over the set $\pi/4 + n\pi$. But this is impossible [the reason for why this is impossible is that it is possible to enumerate the set $\pi/4 + n\pi$, but it is impossible to enumerate all points in the interval $(0, \infty)$].

**Problem 4.** Let $\vec{r}(t) = \langle e^t, e^t \cos t, e^t \sin t\rangle$. Find the arclength of the curve between the points $(1,1,0)$ and $(e^{\pi/2}, 0, e^{\pi/2})$.

Solution: We find

$$\vec{r}\,'(t) = \langle e^t, e^t(\cos t - \sin t), e^t(\sin t + \cos t)\rangle,$$

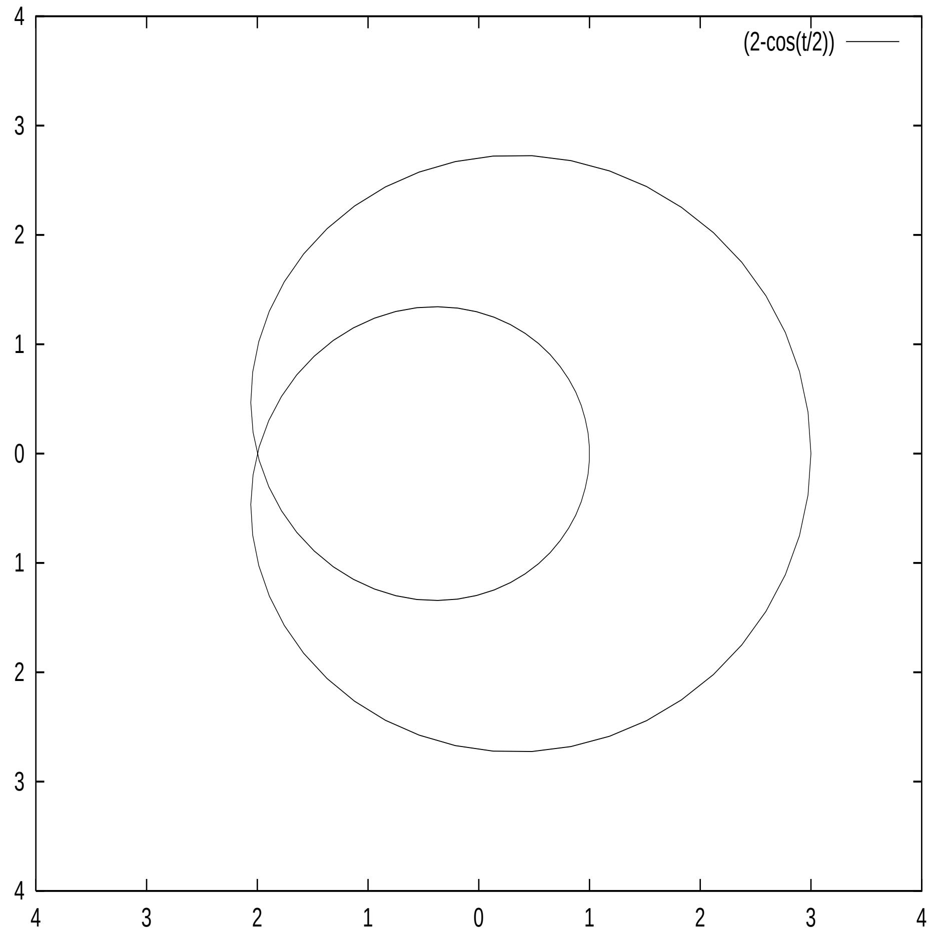
so that

$$\begin{aligned}
\|\vec{r}\,'(t)\| &= e^t\sqrt{1 + (\cos t - \sin t)^2 + (\sin t + \cos t)^2} \\
&= e^t\sqrt{1 + \cos^2 t - 2\sin t\cos t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t} \\
&= e^t\sqrt{1+2} = e^t\sqrt{3}.
\end{aligned}$$

The curve passes through $(1,1,0)$ at $t = 0$ and through $(e^{\pi/2}, 0, e^{\pi/2})$ at $t = \pi/2$. Thus the length is

$$L = \int_0^{\pi/2} \|\vec{r}\,'(t)\| dt = \int_0^{\pi/2} e^t\sqrt{3} dt = \sqrt{3}(e^{\pi/2} - 1).$$

**Problem 5.** Sketch the polar curve $r = 2 - \cos \frac{q}{2}$.



**Problem 6.** Let $S$ be the unit sphere $x^2 + y^2 + z^2 = 1$. Find the equations describing $S$ in (a) spherical coordinates and (b) cylindrical coordinates.

Solution: In spherical coordinates, the equation of the sphere of radius 1 is $\mathrm{r} = 1$. In cyllindrical coordinates, the equation is $z^2 + r^2 = 1$ (since the distance from the origin to the point with cylindrical coordinates $(r, \mathrm{q}, z)$ is $\sqrt{z^2 + r^2}$).

**Problem 7.** Let $\vec{u} = \langle 1,2,3\rangle$, $\vec{v} = \langle 4,5,6\rangle$. (a) Find an equation of the plane parallel to both $\vec{u}$ and $\vec{v}$ and passing through the point $(1,-1,0)$. (b) Find a unit normal vector to that plane. (c) Find the angle between the vectors $\vec{u}$ and $\vec{v}$.

Solution: (a) To find a normal vector $\vec{n}$ to the plane, we take the cross product of the given vectors:

$$\begin{aligned}\vec{n} &= \langle 1,2,3\rangle \times \langle 4,5,6\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \vec{i}(-3) - \vec{j}(-6) + \vec{k}(-3). \\ &= \langle -3,6,-3\rangle\end{aligned}$$

Thus an equation of the plane is

$$-3(x-1)+6(y+1)-3z=0.$$

(b) We make $\vec{n}$ into a unit vector by dividing by its length, which is

$$\|\vec{n}\| = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}.$$

So the vector $\langle \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\rangle$ works (as does the negative of it).

(c) We find

$$\begin{aligned}\|\vec{u}\| &= \sqrt{1+4+9} = \sqrt{14} \\ \|\vec{v}\| &= \sqrt{16+25+36} = \sqrt{77} \\ \vec{u}\cdot\vec{v} &= 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32\end{aligned}$$

Thus if we denote the angle between the vectors by $\theta$ we find that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{32}{\sqrt{14\cdot 77}} = \frac{32}{7\sqrt{22}}.$$

Thus

$$\theta = \cos^{-1}\frac{32}{7\sqrt{22}}.$$

**Problem 8.** True or False. For each of the following statements, write whether it is true or false for the indicated choices of vectors.

| Statement | True/False |
|---|---|
| $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ for all $\vec{u}$ and $\vec{v}$ | True |
| $\vec{u}\times\vec{v}=\vec{v}\times\vec{u}$ for all $\vec{u}$ and $\vec{v}$ | False |
| $(\vec{v}\cdot\vec{u})\vec{w}=\vec{v}(\vec{u}\cdot\vec{w})$ for all $\vec{u},\vec{v}$ and $\vec{w}$ | False |
| $\vec{u}\cdot\vec{v}=0$ if $\vec{u}$ and $\vec{v}$ are orthogonal | True |
| $\vec{u}\times\vec{v}=0$ if $\vec{u}$ and $\vec{v}$ are orthogonal | False |
| $(\vec{u}\times\vec{v})\times\vec{w}$ is the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$ | False |
| If two vectors are orthogonal, the projection of one of them onto the other is 0 | True |
| If $\vec{u}$ and $\vec{v}$ lie in the $(y,z)$ plane, $\vec{u}\times\vec{v}$ is parallel to $\vec{i}$ | True |
| If $\vec{v}\cdot\vec{w}=0$ for all $\vec{w}$, then $\vec{v}$ must be zero | True |
| If $\vec{v}\cdot\langle 1,0,0\rangle=\Vert\vec{v}\Vert$, then $\vec{v}$ is parallel to $\langle 1,0,0\rangle$ | True |

Notes: A counterexample to #3 would be to take the vectors so that $\vec{v}$ and $\vec{u}$ are perpendicular, but $\vec{u}$ and $\vec{w}$ are not. In #4, compute the angle between the two vectors. In #5, the cross product is zero if the vectors are parallel. In #6 the result of the triple cross product is a vector, not a volume; take the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ to get the triple product to be zero (while the associated volume is 1). In #8, the cross product is perpendicular to the plane containing the two vectors, and hence parallel to the $x$-axis. In #9, Take $\vec{w}$ to be the vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ to conclude that all components of $\vec{v}$ must be zero. In #10, compute the angle between the vector $\vec{v}$ and $\langle 1,0,0\rangle$ to conclude that its cosine is 1, so that the vectors are parallel.

**Problem 9.** Let $f(x,y) = x^3 + \cos(xy) + 2xy^2$. (a) Find the gradient of $f$. (b) Find the directional derivative of $f$ in the direction of $\overrightarrow{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. (c) Find the equation of the tangent plane to the graph of $f(x,y)$ at the point $x = 0$, $y = 2$.

Solution: (a) Ñ$f = \langle 3x^2 - y\sin(xy) + 2y^2, -x\sin(xy) + 4xy\rangle$. (b) The directional derivative is

$$Ñf \cdot \overrightarrow{u} = \frac{1}{\sqrt{2}}(3x^2 - y\sin(xy) + 2y^2 - x\sin(xy) + 4xy).$$

(c) The normal vector is given by $\langle f_x, f_y, -1\rangle = \langle 8, 0, -1\rangle$. The value of $f$ at $(0,2)$ is 1. So the point $(0,2,1)$ is on the tangent plane. The desired equation is

$$8x - (z - 1) = 0.$$

**Problem 10.** If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, (a) use the chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.
(b) Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Solution: By the chain rule, we have

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta \\
&\quad + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.
\end{aligned}$$

**Problem 11.** Find all critical points of the function

$$f(x,y) = x^3 - 3x + y^4 - 2y^2$$

and classify them as local minima, maxima or saddle points.

Solution: We find $f_x = 3x^2 - 3$, $f_y = 4y^3 - 4y$. To be at a critical point, we must have $f_x = 0$ and $f_y = 0$, i.e.

$$\begin{aligned} 3x^2 &= 3 \\ 4y(y^2-1) &= 0. \end{aligned}$$

Thus the critical points are when $x = \pm 1$ and $y = 0$, or $y = \pm 1$. There are 6 critical points: $(-1,0)$, $(1,0)$, $(-1,1)$, $(1,1)$, $(-1,-1)$ and $(1,-1)$.

We compute $D = f_{xx} f_{yy} - f_{xy} f_{yx} = 6x \cdot 4(3y^2 - 1) = 24x(3y^2 - 1)$. Also, $f_{xx} = 6x$. Thus

| Point | $D$ | $f_{xx}$ | Conclusion |
|---|---|---|---|
| $(-1,0)$ | $> 0$ | $< 0$ | Local max |
| $(1,0)$ | $< 0$ | | Saddle |
| $(-1,1)$ | $< 0$ | | Saddle |
| $(1,1)$ | $> 0$ | $> 0$ | Local min |
| $(-1,-1)$ | $< 0$ | | Saddle |
| $(1,-1)$ | $> 0$ | $> 0$ | Local min |

**Problem 12.** Find the maximum of the function

$$f(x,y) = e^{-xy}$$

in the region $x^2 + 4y^2 \leq 1$.

Solution: We first check the inside of the region. To be at a critical point, we must have $f_x = 0$ and $f_y = 0$. Now, $f_x = -ye^{-xy}$ and $f_y = -xe^{-xy}$. So the only critical point is $x = y = 0$, and the value of the function at this point is 1.

We now check the boundary. Using Lagrange multipliers with the constraint equation $x^2 + 4y^2 = 1$ we get

$$\begin{aligned} -ye^{-xy} &= 2\lambda x \\ -xe^{-xy} &= 8\lambda y \\ x^2 + 4y^2 &= 1. \end{aligned}$$

We note first that $x$ cannot be zero. Indeed, if it were, we would get that $y = 0$ from the first equation (note that $e^{-xy}$ is never zero), and $x = y = 0$ contradicts the last equation. For the same reason, $y$ cannot be zero. Lastly, it cannot happen that $\lambda = 0$ (otherwise we get $x = y = 0$ again). So $x, y, \lambda$ are all different from zero. We can thus divide the first equation by the second to get

$$\frac{y}{x} = \frac{2}{8}\frac{x}{y},$$

which shows that $x^2 = 4y^2$. Thus from the last equation we get $8y^2 = 1$, so that $y = \pm\frac{1}{2\sqrt{2}}$. Therefore, we get the points

$$(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

The values of the function at these points are, respectively, $e^{\frac{1}{4}}$, $e^{-\frac{1}{4}}$, $e^{-\frac{1}{4}}$ and $e^{\frac{1}{4}}$. Since $e > 1$, the largest value of the funciton on the boundary is $e^{\frac{1}{4}}$. Since this is bigger than 1, this is the maximum of the function on the given region.

**Problem 13.** Use the method of Lagrange multipliers to find the maximum of

$$f(x,y,z) = xyz$$

subject to the constraint $x^2+y^2+z^2 = 1$.

We get the following system of equations:

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ yz &= 2\lambda z \\ x^2+y^2+z^2 &= 1. \end{aligned}$$

Since if any of $x,y,z$ are zero, the value of $f(x,y,z)$ is zero, and the maximal value of $f$ on the given set is clearly bigger than zero (just take any $x,y,z$ nonzero positive with $x^2+y^2+z^2 = 1$), we may assume that $x,y,z$ are nonzero. Multiplying the first equation by $x$, the second by $y$ and the third by $z$ gives us the equations:

$$\begin{aligned} xyz &= 2\lambda x^2 \\ xyz &= 2\lambda y^2 \\ xyz &= 2\lambda z^2 \\ x^2+y^2+z^2 &= 1. \end{aligned}$$

Note that $\lambda$ cannot equal to zero (otherwise $xyz = 0$ from the first equation, which cannot happen at a point of maximum of our function). Thus the first two equations combined give

$$2\lambda x^2 = 2\lambda y^2$$

and since $\lambda \neq 0$, $x^2 = y^2$. Similarly, we get from the second and third equations that $y^2 = z^2$. Thus $x^2 = y^2 = z^2$. From the last equation we get that $3x^2 = 1$, so that $x = \pm 1/\sqrt{3}$. There are all together 8 solutions $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ (signs uncoordinated). The value of the funciton at these points is $\pm 1/(3\sqrt{3})$. It follows that the maximal value of the function is $1/(3\sqrt{3})$.