SOLUTIONS

MATH 31B LECTURE 1 AND 3
2ND MIDTERM VERSION B

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of your professor within 2 weeks of the exam.*

#1	#2	#3	#4	#5	#6	Total

Section 1__ 3__ meets: Tuesday Thursday

TA name: ____________

SID: ____________

Name: CHRISTINA ORAN

Problem 1. (Multiple choice, 10 pts) Evaluate the integral

$$\int_0^{e-1} \frac{x^2+2x+3}{(x^2+1)(x+1)} dx$$

Indicate your answer in the box below:

$\boxed{B}$ (a) $\tan^{-1}(e-1)+1$; (b) $2\tan^{-1}(e-1)+1$; (c) $\tan^{-1}(e-1)+3$ (d) $\tan^{-1}(e-1)-3$; (e) None of the above.

$$\frac{x^2+2x+3}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} = \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$Ax^2 + Ax + Bx + B + Cx^2 + C = x^2+2x+3$$

$$\begin{cases} A+C=1 \\ A+B=2 \\ B+C=3 \end{cases} \Rightarrow \begin{cases} B-C=1 \\ B+C=3 \\ A+C=1 \end{cases} \Rightarrow 2B=4,\ B=2$$

$$C=1$$
$$A=0$$

$$\int_0^{e-1} \frac{x^2+2x+3}{(x^2+1)(x+1)} dx = \int_0^{e-1}\left(\frac{2}{x^2+1} + \frac{1}{x+1}\right)dx = \left(2\tan^{-1}x + \ln|x+1|\right)\Big|_0^{e-1}$$

$$= 2\tan^{-1}(e-1) + \ln e - 2\tan^{-1} 0 - \ln 1$$
$$= 2\tan^{-1}(e-1) + 1$$

Problem 2. (Multiple choice, 10 pts) Find the length of the curve $y = \frac{2}{3}x^{3/2}$, $0 \le x \le 2$.

$\boxed{D}$ (a) ∞; (b) $\frac{14}{3}$; (c) $\frac{(3\sqrt{3}-1)}{2}$; (d) $\frac{2}{3}(3\sqrt{3}-1)$; (e) None of the above.

$$y = \frac{2}{3}x^{3/2}$$

$$\frac{dy}{dx} = x^{1/2} \qquad \left(\frac{dy}{dx}\right)^2 = x$$

$$L = \int_0^2 \sqrt{1+x}\,dx = \int_1^3 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^3 = \frac{2}{3}(3^{3/2}-1)$$

$$= \frac{2}{3}(3\sqrt{3}-1)$$

$u = 1+x$
$du = dx$

Problem 3. (Multiple choice, 10 pts) Let the sequence $\{a_n\}$ be defined by: $a_n = n^{2/n}$. Find the limit $\lim_{n\to\infty} a_n$, if the limit exists. Indicate your answer in the box below:

B $\boxed{}$ (a) 0; (b) 1; (c) π; (d) limit does not exist; (e) None of the above.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} e^{\ln(n^{2/n})} = \lim_{n\to\infty} e^{\frac{2}{n}\ln n}$$

$$= e^{\lim_{n\to\infty} \frac{2}{n}\ln n}$$ if the limit in the exponent exists.

$$\lim_{n\to\infty} \frac{2}{n}\ln n = 2\lim_{n\to\infty} \frac{\ln n}{n} \underset{\text{L'Hôpital}}{=} 2\lim_{n\to\infty} \frac{1/n}{1} = 2\lim_{n\to\infty} \frac{1}{n} = 0$$

so $\lim_{n\to\infty} a_n = e^0 = 1$

Problem 4. (25 pts) Consider the series $\sum_{n=3}^{\infty} \frac{1}{3^n}$.

(a) Determine if the series is convergent;
(b) If the series is convergent, find its limit.

$$\sum_{n=3}^{\infty} \frac{1}{3^n} = \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^{n-3} = \frac{1}{27} \sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^{n-3}$$

$$= \frac{1}{27} \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1} = \frac{1}{27} \frac{1}{1-\frac{1}{3}} = \frac{1}{27-9} = \frac{1}{18}$$

let $k=n-2$

Geometric series, $r = \frac{1}{3}$

Problem 5. (20 pts) Determine if the series $\sum_{n=1}^{\infty} ne^{-2n}$ is convergent or divergent. Justify your answer by referring to a theorem from the course.

$$\sum_{n=1}^{\infty} ne^{-2n} = \sum_{n=1}^{\infty} \frac{n}{e^{2n}}$$

This sequence (ne^{-2n}) is positive and decreasing, with $f(x) = xe^{-2x}$ continuous, so we can apply the integral test for convergence:

$\int_1^{\infty} xe^{-2x}\,dx$ converges iff $\sum_{n=1}^{\infty} ne^{-2n}$ converges.

$$\int_1^{\infty} xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_1^{\infty} + \frac{1}{2}\int_1^{\infty} e^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_1^{\infty} - \frac{1}{4}e^{-2x}\Big|_1^{\infty}$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) + \frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}$$

$u = x \quad dv = e^{-2x}dx$

$du = dx \quad v = -\frac{1}{2}e^{-2x}$

$$= -\lim_{t\to\infty}\frac{t}{2e^{2t}} - 0 + \frac{3}{4}\cdot\frac{1}{e^2}$$

$$= -\lim_{t\to\infty}\frac{1}{2\cdot 2e^{2t}} + \frac{3}{4e^2} = \frac{3}{4e^2}$$

So since the improper integral converges, so does the series.

Problem 6. (25 pts) Let $a_n = \dfrac{n!}{(3n)!}$. Determine if the sequence a_n is convergent, and if it is, find its limit.

$$a_n = \frac{n!}{(3n)!} = \frac{n!}{(3n)(3n-1)\cdots(n+1)\cdot n!} = \frac{1}{(3n)(3n-1)\cdots(n+2)(n+1)}$$

So for $n \geq 1$,

$$0 \leq a_n \leq \frac{1}{3n}$$

$\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{3n} = 0$, so by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

it converges with