MATH 31B LECTURE 1 AND 3
PRACTICE MIDTERM

Solutions.

Problem 1. (Multiple choice, 10 pts) Evaluate the improper integral

$$\int_{-1}^{1} \frac{1}{x} dx$$

Indicate your answer in the box below:

$\boxed{D}$ (a) 1; (b) 0; (c) -1; (d) integral diverges; (e) None of the above.

$$\int_{-1}^{1} \frac{1}{x} dx = \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x} dx + \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x} dx$$

$$= \lim_{t \to 0^-} \ln|x| \Big|_{-1}^{t} + \lim_{t \to 0^+} \ln|x| \Big|_{t}^{1}$$

Since $\lim_{t \to 0^-} \ln|t|$ doesn't exist, the integral diverges.

Problem 2. (Multiple choice, 10 pts) Find the length of the curve $y = 2\cosh(x)$, $0 \le x \le 1$.

$\boxed{E}$ (a) $\sinh(1)$; (b) e; (c) $2\sinh(1)\pi$; (d) $2e$; (e) None of the above.

$$L = \int_0^1 \sqrt{1 + 4\sinh^2 x}\, dx$$

$$> \int_0^1 \sqrt{1+\sinh^2 x}\, dx = \int_0^1 \cosh x\, dx$$

$$= \sinh 1 = \frac{1}{2}(e - e^{-1}).$$

So answer is $> \sinh 1$, so (a) is not the answer.

$$\int_0^1 \sqrt{1+4\sinh^2 x}\, dx < \int_0^1 \sqrt{4 + 4\sinh^2 x}\, dx = \int_0^1 2\cosh x\, dx$$

$$= 2\sinh 1$$

So answer is $< 2\sinh 1$.

Now $2\sinh(1)\cdot\pi$ and $2e$ are $> 2\sinh 1$. So (c), (d) cannot be.

$2\sinh(1) = 2\left(\frac{1}{2}e - e^{-1}\right) < e$ So $e > 2\sinh 1$.

So it's not (b). Thus (E).

Problem 3. (Multiple choice, 10 pts) Let the sequence $\{a_n\}$ be defined by: $a_1 = 1$, $a_2 = 0$, $a_{n+1} = \frac{1}{2}(\sin(a_n) + \sin(a_{n-1}))$. Assume that it is known that a_n converges to a finite limit L as $n \to \infty$. Determine L. Indicate your answer in the box below:

$\boxed{A}$ (a) 0; (b) $\sin(1)$; (c) π; (d) cannot be determined; (e) None of the above.

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}\left(\sin(a_{n-1}) + \sin(a_{n-2})\right)$$

since $a_n = \frac{1}{2}\left(\sin(a_{n-1}) + \sin(a_{n-2})\right)$.

So $$L = \frac{1}{2}\left(\lim_{n\to\infty} \sin(a_{n-1}) + \lim_{n\to\infty} \sin(a_{n-2})\right)$$

$$= \frac{1}{2}\left(\sin \lim_{n\to\infty} a_{n-1} + \sin \lim_{n\to\infty} a_{n-2}\right)$$

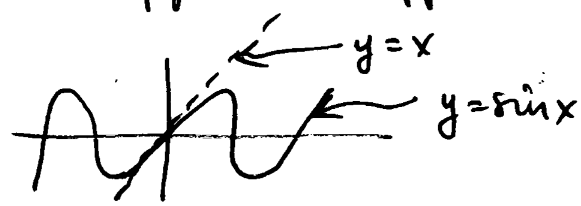
because sin is continuous.

So $L = \frac{1}{2}(\sin L + \sin L) = \sin L$.

Hence $L = \sin L$.

Now this equation has a unique solution, $L = 0$.

(this is ~~apparent~~ apparent from the graph:)



Problem 4. (25 pts) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(a) Determine if the series is convergent;
(b) If the series is convergent, find its limit.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

So $$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{N} - \frac{1}{N+1}$$

$$= 1 - \frac{1}{N+1}.$$

Hence $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N \to \infty} 1 - \frac{1}{N+1} = 1.$$

Series converges to 1.

Problem 5. (25 pts) Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent or divergent.

$\frac{1}{x^2+1}$ is decreasing and

$$\int_1^\infty \frac{1}{x^2+1} = \lim_{t\to\infty} \tan^{-1} t - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty.$$

Series converges by the integral test.

Problem 6. (20 pts) Let $f(x) = \sin(x)$. Find an n so that the trapezoidal rule approximates the integral $\int_0^1 \sin(x)dx$ with an error of no more than $1/120$.

The error involved is $\leq \dfrac{K(b-a)^3}{12n^2}$

where $K \geq |f''(x)|$ for all $a \leq x \leq b$.

Here $a=0$, $b=1$, $f(x) = \sin(x)$, $f''(x) = -\sin x$.

Thus max of $|f''(x)|$ on $[0,1]$ is $\sin(1)$.

So we can take K to be any number $\geq \sin(1)$, e.g. $K=1$. Thus we want

$$\frac{K(1-0)^3}{12n^2} < \frac{1}{120}, \text{ so } \frac{1}{12n^2} < \frac{1}{120},$$

so $12n^2 > 120$, $n^2 > 10$, so $n \geq 4$.

(Note: we could use $K = 0{,}841471$ since $\sin(1) \approx 0.84147098\ldots$ Then we get:

$$\frac{0{,}841471}{12n^2} < \frac{1}{120}, \text{ so } 12n^2 > 120 \cdot 0{,}841471,$$

$n^2 > 8{,}41471$, so $n \geq 3$.

Using a better K (which is harder to compute!) gave us a marginal improvement in n).

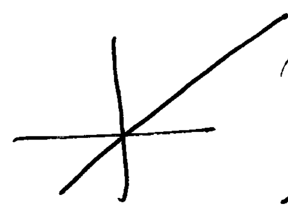
Problem 7. (20 pts) Determine if the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is convergent or divergent.

$$\frac{n^2}{2^n} = \frac{n^2}{(\sqrt{2})^n} \cdot \frac{1}{(\sqrt{2})^n}.$$

Now $\ln \frac{n^2}{(\sqrt{2})^n} = 2\ln n - \frac{n}{2}\ln 2 < 0$

for n large enough (this follows from inspecting the graphs of $2\ln n$

and $\frac{n}{2}\ln 2$



). So for n large, $\frac{n^2}{(\sqrt{2})^n} < 1$.

So for n large enough, $0 < \frac{n^2}{2^n} \leq \frac{1}{(\sqrt{2})^n} \cdot \frac{n^2}{(\sqrt{2})^n} \leq \frac{1}{(\sqrt{2})^n}$.

Since $\sum \frac{1}{(\sqrt{2})^n}$ is a geometric series with $r = \frac{1}{\sqrt{2}} < 1$, it converges. From this it follows that

$\sum \frac{n^2}{2^n}$ is finite (since if it were ∞, we would get that $\sum \frac{1}{(\sqrt{2})^n}$ were also infinite, which is impossible).

Problem 8. (20 pts) Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is convergent or divergent.

Since ~~x~~ and log ~~x~~ are both increasing,

$\frac{1}{x \ln x}$ is decreasing. Also $\frac{1}{x \ln x} \geq 0$ if $x \geq 2$.

We apply the integral test

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x}\, dx \qquad u = \ln x, \quad du = \frac{dx}{x}, \quad x = e^u$$

$$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\, du$$

$$= \lim_{t \to \infty} \ln u \Big|_{\ln 2}^{\ln t}$$

$$= \lim_{t \to \infty} \ln \ln t - \ln \ln 2$$

$$= \infty$$

So the series diverges.