Problem 1. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x\to 1} \frac{x-1}{\ln(x^2)}.$$

Indicate your answer in the box below:

$\boxed{B}$ (a) 1; (b) $\frac{1}{2}$; (c) 2; (d) ∞; (e) None of the above.

$$\lim_{x\to 1} \frac{x-1}{\ln x^2} = \lim_{x\to 1} \frac{1}{\frac{2x}{x^2}} = \lim_{x\to 1} \frac{x}{2} = \frac{1}{2}.$$

Problem 2. (Multiple choice, 20 pts) Let $f(x) = 4x - \cos x$. Find the value of the derivative $(f^{-1})'(2\pi)$. Indicate your answer in the box below:

$\boxed{B}$ (a) 1/3; (b) 1/5; (c) 2π; (d) 0; (e) None of the above.

Notice that $f(\pi/2) = 4 \cdot \pi/2 - \cos(\pi/2) = 2\pi - 0 = 2\pi$.

So $f^{-1}(2\pi) = \pi/2$.

$$(f^{-1})'(2\pi) = \frac{1}{f'(f^{-1}(2\pi))} = \frac{1}{f'(\pi/2)}.$$

$f'(x) = 4 + \sin x$, so $(f^{-1})'(2\pi) = \frac{1}{f'(\pi/2)} = \frac{1}{4 + \sin \pi/2} = \frac{1}{5}$.

Problem 3. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x\to 0}(1+2x)^{\frac{2}{x}}.$$

Indicate your answer in the box below:

$\boxed{E}$ (a) 1; (b) ∞; (c) 4; (d) 9; (e) None of the above.

$$\lim_{x\to 0}(1+2x)^{2/x} = \lim_{x\to 0} e^{\ln(1+2x)^{2/x}} = \lim_{x\to 0} e^{\frac{2\ln(1+2x)}{x}}$$

$$= e^{\lim_{x\to 0}\frac{2\ln(1+2x)}{x}}$$

By L'Hospital's rule, $\lim_{x\to 0}\frac{2\ln(1+2x)}{x} = \lim_{x\to 0}\frac{4}{1+2x} = 4.$

So $\lim_{x\to 0}(1+2x)^{2/x} = e^4.$

Problem 4. (25 pts) Use integration by parts to evaluate the definite integral

$$\int_0^1 e^x \cos x \, dx$$

$$\int_0^1 e^x \cos x \, dx = e^x \cos x \Big|_0^1 - \int_0^1 e^x (-\sin x) \, dx$$

Parts: $u = \cos x$ $dv = e^x dx$
$du = -\sin x \, dx$ $v = e^x$

$$= e^1 \cos 1 - e^0 \cos 0 + \int_0^1 e^x \sin x \, dx$$

Parts: $u = \sin x$, $dv = e^x dx$
$du = \cos x \, dx$, $v = e^x$

$$= e \cos 1 - 1 + \left[e^x \sin x \Big|_0^1 - \int_0^1 e^x \cos x \, dx \right]$$

$$= e \cos 1 - 1 + e \sin 1 - e^0 \sin 0 - \int_0^1 e^x \cos x \, dx$$

$$= e(\cos 1 + \sin 1) - 1 - \int_0^1 e^x \cos x \, dx.$$

So if $I = \int_0^1 e^x \cos x \, dx$, then

$$I = e(\cos 1 + \sin 1) - 1 - I, \text{ so}$$

$$2I = e(\cos 1 + \sin 1) - 1,$$

$$I = \frac{1}{2} e (\cos 1 - \sin 1) - \frac{1}{2}.$$

Problem 5. (35 pts) Let $f(x) = xe^{-x}$. Sketch the graph of f. Be sure to indicate convexity, asymptotes and extreme points.

Domain: $(-\infty, \infty)$

Asymptotes: $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

$\lim_{x\to-\infty} xe^{-x} = -\infty.$

Intersepts: $f(0)=0$; $f(x)=0$ means $xe^{-x}=0$, so $x=0$ or $e^{-x}=0$ (has no solution).

So $f(x)=0$ only for $x=0$.

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$;

$f'(x) > 0$ if $x<1$, increasing

$f'(x) < 0$ if $x>1$, decreasing.

1 is a maximum.

$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(x-2).$

$f''>0$ if $x>2$, concave up

$f''<0$ if $x<2$, concave down.