Problem 1. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x\to 1} \frac{\ln(x^2)}{x-1}.$$

Indicate your answer in the box below:

$\boxed{c}$ (a) 1; (b) $\frac{1}{2}$; (c) 2; (d) ∞; (e) None of the above.

$$\lim_{x\to 1} \frac{\ln(x^2)}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x^2}\cdot 2x}{1} = \lim_{x\to 1} \frac{2}{x} = 2.$$

Problem 2. (Multiple choice, 20 pts) Let $f(x) = 4x + \cos x$. Find the value of the derivative $(f^{-1})'(2\pi)$. Indicate your answer in the box below:

$\boxed{a}$ (a) $1/3$; (b) $1/5$; (c) 2π; (d) 0; (e) None of the above.

Notice that

$$f(\pi/2) = 4 \cdot \pi/2 + \cos(\pi/2) = 2\pi + 0 = 2\pi.$$

So $f^{-1}(2\pi) = \pi/2.$

Thus

$$(f^{-1})'(2\pi) = \frac{1}{f'(f^{-1}(2\pi))} = \frac{1}{f'(\pi/2)}.$$

Since $f'(x) = 4 - \sin x$, we get

$$(f^{-1})'(2\pi) = \frac{1}{f'(\pi/2)} = \frac{1}{4 - \sin \pi/2} = \frac{1}{4-1} = \frac{1}{3}.$$

Problem 3. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x\to 0}(1+3x)^{\frac{3}{x}}.$$

Indicate your answer in the box below:

$\boxed{E}$ (a) 1; (b) ∞; (c) 4; (d) 9; (e) None of the above.

$$\lim_{x\to 0}(1+3x)^{\frac{3}{x}} = \lim_{x\to 0} e^{\ln(1+3x)^{\frac{3}{x}}}$$

$$= \lim_{x\to 0} e^{\frac{3\ln(1+3x)}{x}}$$

$$= e^{\lim_{x\to 0}\frac{3\ln(1+3x)}{x}}$$

Using L'Hospital's rule, $\lim_{x\to 0}\frac{3\ln(1+3x)}{x} = \lim_{x\to 0}\frac{\frac{9}{1+3x}}{1} = 9.$

Hence $\lim_{x\to 0}(1+3x)^{\frac{3}{x}} = e^9.$

Problem 4. (25 pts) Use integration by parts to evaluate the definite integral

$$\int_0^1 e^x \sin x \, dx$$

$$\int_0^1 e^x \sin x \, dx = e^x \sin x \Big|_0^1 - \int_0^1 e^x \cos x \, dx$$

Parts, $u = \sin x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = e^x$

$$= e \sin 1 - e \sin 0 - \int_0^1 e^x \cos x \, dx$$

Parts, $u = \cos x, \quad dv = e^x dx$

$du = -\sin x \, dx, \quad v = e^x$

$$= e \sin 1 - \left(e^x \cos x \Big|_0^1 - \int_0^1 e^x (-\sin x) \, dx \right)$$

$$= e \sin \cancel{1} - e \cos 1 + e^0 \cos 0 - \int_0^1 e^x \sin x \, dx$$

$$= e (\sin 1 - \cos 1) + 1 - \int_0^1 e^x \sin x \, dx.$$

So if $I = \int_0^1 e^x \sin x \, dx$, then

$$I = e(\sin 1 - \cos 1) + 1 - I, \text{ so}$$

$$2I = e(\sin 1 - \cos 1) + 1 \quad \text{and}$$

$$I = \tfrac{1}{2} e (\sin 1 - \cos 1) + \tfrac{1}{2}.$$

Problem 5. (35 pts) Let $f(x) = xe^x$. Sketch the graph of f. Be sure to indicate convexity, asymptotes and extreme points.

Domain: $(-\infty, \infty)$.

Asymptote: $\lim_{x\to\infty} xe^x = \infty$; $\lim_{x\to-\infty} xe^{+x} = \lim_{x\to-\infty} \frac{x}{e^{-x}}$

$$= \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0.$$

Intercepts: $f(0)=0$; $f(x)=0 \Rightarrow xe^x=0$, so $x=0$ or $e^x=0$ (which has no solutions).

so $f(x)=0$ only for $x=0$.

$f'(x) = e^x + xe^x = e^x(1+x)$; $f'>0$ if $x>-1$, incr. $f'<0$ if $x<-1$ decr.

-1 is a minimum.

$f''(x) = e^x + e^x + xe^x = e^x(2+x)$

$f''>0$ if $x>-2$, concave up;

$f''<0$ if $x<-2$, concave down: