MATH 31B LECTURE 1 AND 3
1ST MIDTERM VERSION P

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of your professor within 2 weeks of the exam.*

The purpose of the practice midterm is to give you a rough idea of the types of problems that you may expect on the actual exam. However, the choice of subjects on the midterm may differ from the actual exam. The practice midterm is not intended to give a representative selection of the subjects covered by the exam. Please consult the class of syllabus for the list of topics that may be covered by the exam.

#1	#2	#3	#4	#5	#6	#7	Total

Section 1__ 3__ meets: Tuesday Thursday

TA name: ______________

SID: ______________

Name: solutions

Problem 1. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}.$$

Indicate your answer in the box below:

$\boxed{A}$ (a) 1; (b) -1; (c) 0; (d) ∞; (e) None of the above.

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \to 0} \frac{\cos x \, e^{\sin x}}{1} = 1$$

Problem 2. (10 pts) Let $f(x) = x^3 + 4$. Compute the derivative of $f^{-1}(x)$.

If $y = f^{-1}(x)$ then $x = y^3 + 4$,

so $y = \sqrt[3]{x-4}$.

$$\frac{d(f^{-1})(x)}{dx} = \frac{d}{dx}(x-4)^{1/3} = \frac{1}{3}(x-4)^{-2/3}.$$

OR

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3f^{-1}(x)^2}$$

$$= \frac{1}{3(\sqrt[3]{x-4})^2} = \frac{1}{3}(x-4)^{-2/3}.$$

Problem 3. (Multiple choice, 10 pts) Evaluate the limit

$$\lim_{x\to 1} x^{\frac{1}{\ln x}}$$

Indicate your answer in the box below:

$\boxed{C}$ (a) 1; (b) ∞; (c) e; (d) $\sqrt{e}$; (e) None of the above.

$$x^{\frac{1}{\ln x}} = e^{\ln x^{\frac{1}{\ln x}}} = e^{\frac{1}{\ln x}\ln x} = e.$$

$$\text{So } \lim_{x\to 1} x^{\frac{1}{\ln x}} = e.$$

Problem 4. (20 pts) Use integration by parts to evaluate the definite integral

$$\int_0^1 \frac{2r^3}{\sqrt{4+r^2}}dr$$

Let $u = r^2$, $dv = \dfrac{2r}{\sqrt{4+r^2}} dr$.

Then $du = 2r\,dr$, $v = 2\sqrt{4+r^2}$

Thus $\displaystyle\int_0^1 \frac{2r^3}{\sqrt{4+r^2}} dr = r^2 \cdot 2\sqrt{4+r^2}\Big|_0^1$

$$- \int_0^1 4r\sqrt{4+r^2}\, dr$$

$$= 2\sqrt{5} - 4\int_0^1 r\sqrt{4+r^2}\, dr.$$

We now substitute $y = 4+r^2$, $dy = 2r\,dr$;

$0 \le r \le 1$ means that $4 \le y \le 5$.

So $\displaystyle\int_0^1 \frac{2r^3}{\sqrt{4+r^2}} dr = 2\sqrt{5} - 2\int_4^5 \sqrt{u}\, du$

$$= 2\sqrt{5} - \frac{2\cdot 2u^{3/2}}{3}\Big|_4^5 = 2\sqrt{5} - \frac{4\cdot 5^{3/2}}{3} + \frac{2\cdot 2\cdot 4^{3/2}}{3}$$

$$= 2\sqrt{5} - \frac{20\sqrt{5}}{3} + \frac{32}{3} = -\frac{14}{3}\sqrt{5} + \frac{32}{3}.$$

Problem 5. (20 pts) Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \tan^3 x dx.$$

$$\int_0^{\pi/4} \tan^3 x\, dx = \int_0^{\pi/4} \tan^2 x \tan x\, dx$$

$$= \int_0^{\pi/4} (\sec^2 x - 1) \tan x\, dx$$

$$= \int_0^{\pi/4} \sec^2 x \tan x\, dx - \int_0^{\pi/4} \tan x\, dx$$

Let $u = \tan x$, $du = \sec^2 x\, dx$;

$0 \le x \le \pi/4$, so $0 \le u \le 1$

$$= \int_0^1 u\, du - \int_0^{\pi/4} \tan x\, dx$$

$$= \frac{1}{2} u^2 \Big|_0^1 - \ln|\sec x| \Big|_0^{\pi/4}$$

$$= \frac{1}{2} - \ln\left(\frac{\sqrt{2}}{2}\right)^{-1} + \ln 1 = \frac{1}{2} + \ln \frac{\sqrt{2}}{2}.$$

$$= \frac{1}{2} + \ln \frac{1}{\sqrt{2}} = \frac{1}{2} - \ln \sqrt{2}$$

$$= \frac{1}{2} - \ln 2^{\frac{1}{2}} = \frac{1}{2} - \frac{1}{2} \ln 2$$

$$= \frac{1}{2}(1 - \ln 2).$$

Problem 6. (30 pts) Let $f(x) = x \ln x$. Sketch the graph of f. Be sure to indicate convexity, asymptotes and extreme points.

Domain: $(0, +\infty)$ since $\ln x$ only defined for $x > 0$.

Asymptotes: $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.$$

$\lim_{x \to \infty} x \ln x = \infty$ since $x \to +\infty$ and $\ln x \to +\infty$.

Derivative: $f'(x) = \ln x + x \cdot \frac{1}{x} = 1 + \ln x$.

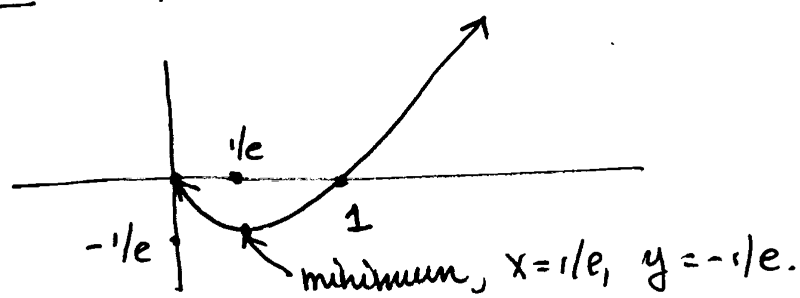
$\ln x > -1$ if $x > 1/e$, $\ln x = -1$ if $x = 1/e$, $\ln x < -1$ if $x < 1/e$.

So: decreasing if $0 < x < 1/e$;
increasing if $1/e < x < \infty$
minimum at $x = 1/e$, $f(1/e) = 1/e \cdot \ln(1/e) = -1/e$.

Convexity: $f''(x) = \frac{d}{dx}(1 + \ln x) = 1/x > 0$ if $x > 0$,

so concave up.

Intercept: $f(x) = 0$ if $x = 0$ (not in domain) or $\ln x = 0$, i.e., if $x = 1$



Problem 7. (30pts) Use a trigonometric substitution to evaluate the integral

$$\int \frac{1}{\sqrt{1+x^2}}dx$$

Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$.

$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{1}{\sec\theta}\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln|\sec\theta + x| + C$$

$$= \ln|\sqrt{1+x^2} + x| + C.$$