12.3 The Integral Test

3-8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Let $f(x) = \frac{1}{x^4}$. Then f is continuous, positive and decreasing. Since $\int_1^{\infty} \frac{1}{x^4}dx = \frac{1}{5}$ the series is converges

5. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

Let $f(x) = \frac{1}{3x+1}$. Then f is continuous, positive and since $f'(x) = \frac{-3}{(3x+1)^2}$ we see that f is decreasing too. Now

$$\int_1^{\infty} \frac{1}{3x+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{3x+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{3} \ln(3x+1) \Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{3}\ln(3t+1) - \frac{1}{3}\ln 4$$

$$= \infty$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges

7. $\sum_{n=1}^{\infty} ne^{-n}$

Let $f(x) = xe^{-x}$. Then f is continuous, positive and since

$$f'(x) = e^{-x} + xe^{-x}\cdot(-1) = e^{-x}(1-x)$$

we see that f is decreasing too. Now

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} xe^{-x}\,dx$$

$$= \lim_{t\to\infty} \left(-xe^{-x} \Big|_1^t + \int_1^t e^{-x}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-xe^{-x} \Big|_1^t - e^{-x} \Big|_1^t \right)$$

$$= \lim_{t\to\infty} \left(-te^{-t} + e^{-1} - e^{-t} + e^{-1} \right)$$

$$= 2e^{-1}$$

Therefore the series $\sum_{n=1}^{\infty} ne^{-n}$ is convergent.

9-24 Determine whether the series is convergent or divergent.

9. $\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$

This is a p-series with $p = 0.85 \leq 1$ and therefore the series is divergent.

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

We recognize that

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

and this is a p-series with $p = 3 > 1$ and therefore the series is convergent

13. $\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3}$

We consider the series

$$\sum_{n=1}^{\infty} \frac{5}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^3}$$

The first is a p-series with $p = 3 > 1$ and therefore is convergent. The second can be written as $\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{2}{n^{5/2}}$ and is a p-series with $p = 5/2$

$\sum^{\infty} 5 - 2\sqrt{n}$

15. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$.

Let $f(x) = \frac{1}{x^2+4}$. Then f is continuous, positive and since $f'(x) = \frac{-2x}{(x^2+4)^2}$ we see that f is decreasing. Now

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{4}\int_1^t \frac{1}{(\frac{x}{2})^2+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{4}\int_{1/2}^{t/2} \frac{2\,du}{u^2+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_{1/2}^{t/2} \frac{du}{u^2+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\arctan u \Big|_{1/2}^{t/2}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\arctan t/2 - \arctan 1/2\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan\frac{1}{2}\right).$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges.

17. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Let $f(x) = \frac{x}{x^2+1}$. Then f is continuous, positive and since $f'(x) = \frac{4-x^2}{(x^2+4)^2}$ we see that f is decreasing on $[2,\infty)$. Now

$$\int_2^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_2^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_5^{t^2+1} \frac{\frac{1}{2}du}{u} = \lim_{t\to\infty} \frac{1}{2}\left(\ln t^2+1 - \ln 5\right) = \infty$$

and therefore the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent.