MATH 31B HW 6

§ 12.1

15. $a_n = n(n-1)$ $\lim_{n\to\infty} a_n = \infty \Rightarrow \boxed{\text{Divergent}}$

17. $a_n = \frac{3+5n^2}{n+n^2}$ $\lim_{n\to\infty} a_n = \frac{3/n^2+5}{1/n+1} = \boxed{5}$

19. $a_n = \frac{2^n}{3^{n+1}}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \boxed{0}$

21. $a_n = \frac{(-1)^{n-1}n}{n^2+1}$ $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = \boxed{0} \Rightarrow \lim_{n\to\infty} a_n = 0$

23. $a_n = \cos(n/2)$ $\lim_{n\to\infty} a_n$ DNE. (does not exist) since cosine is an oscillating (a periodic) function $\Rightarrow \boxed{\text{Divergent}}$

25. $\left\{\frac{(2n-1)!}{(2n+1)!}\right\}$ For $n \geq 1$ $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)} = \frac{1}{2n(2n+1)}$

$$\Rightarrow \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{1}{2n(2n+1)} = \boxed{0}$$

27. $\left\{\frac{e^n+e^{-n}}{e^{2n}-1}\right\}$ $\lim_{n\to\infty} \frac{e^n+e^{-n}}{e^{2n}-1} = \lim_{n\to\infty} \frac{1+e^{-2n}}{e^n-e^{-n}} = \boxed{0}$

29. $\{n^2e^{-n}\}$ $\lim_{n\to\infty} n^2e^{-n} = \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = \boxed{0}$

31. $a_n = \frac{\cos^2 n}{2^n}$ For all $n \in \mathbb{Z}^+$, $-1 \leq \cos n \leq 1 \Rightarrow 0 \leq \cos^2 n \leq 1$

$$\Rightarrow 0 = \lim_{n\to\infty} \frac{0}{2^n} \leq \lim_{n\to\infty} \frac{\cos^2 n}{2^n} \leq \lim_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\cos^2 n}{2^n} = \boxed{0}$$

33. $a_n = n\sin(1/n)$ $\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{m\to 0} \frac{\sin(m)}{m}$ (let $m = 1/n$)

$$= \boxed{1}$$

35. $a_n = \left(1+\frac{2}{n}\right)^{1/n}$ $\lim_{n\to\infty} a_n = \boxed{1}$

let $b = \left(1+\frac{2}{n}\right)^{1/n}$, $\ln b = \frac{\ln\left(1+\frac{2}{n}\right)}{n}$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{n} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}}(-2n^{-2}) = \lim_{n\to\infty} \frac{-2}{n^2+2n} = 0$$

so, $n \to \infty$, $\ln b \to 0 \Rightarrow b \to e^0 = 1$

37. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \cdots\}$ let a_n be this sequence

$\lim_{n\to\infty} a_n$ D.N.E. since the sequence oscillates between 0 and 1

9. $a_n = \frac{n!}{2^n}$ Note that both the top and bottom have n terms

$$a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \cdots (n-2)\cdot(n-1)\cdot n}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 \cdots 2 \cdot 2\cdot 2}$$ > (1/2) (n/2) = n/4, since k/2>1 for k>2.

Since n/4 tends to infinity as n tends to infinity, the series is divergent

$\Rightarrow \boxed{\text{Divergent}}$

14. $a_n = \frac{1}{5^n}$

$a_{n+1} - a_n = \frac{1}{5^{n+1}} - \frac{1}{5^n} = \frac{1}{5^n}\left(\frac{1}{5} - 1\right) < 0$ $\Rightarrow a_n$ is monotonically decreasing

Since $a_n = \left(\frac{1}{5}\right)^n$ and a_n is decreasing, we have $0 \le a_n \le 1$ $\forall n \in \mathbb{Z}^+$

$\Rightarrow a_n$ is bounded

55. $a_n = \frac{1}{2n+3}$

$a_{n+1} - a_n = \frac{1}{2n+5} - \frac{1}{2n+3} = \frac{-2}{(2n+5)(2n+3)} < 0$ for all $n > 0$

$\Rightarrow a_n$ is decreasing

$n > 0 \Rightarrow a_n > 0$, a_n decreases $\Rightarrow a_n \le a_1 = \frac{1}{5}$

$\Rightarrow 0 \le a_n \le 1/5$ $\Rightarrow a_n$ is bounded

51. $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$

let a_n denote this sequence.

Note. $a_1 = \sqrt{2} = 2^{1/2}$, $a_2 = \sqrt{2\sqrt{2}} = \sqrt{2}\cdot\sqrt{\sqrt{2}} = 2^{1/2}\cdot 2^{1/4} = 2^{1/2+1/4}$

$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2}\cdot\sqrt{\sqrt{2}}\cdot\sqrt{\sqrt{\sqrt{2}}} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$, $\cdots$

so. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\cdots+(\frac{1}{2})^n}$, $\Rightarrow \log_2 a_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n$

let $n \to \infty$, $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \to \frac{1}{2}\frac{1}{1-\frac{1}{2}} = 2/2 = 1$ (geometric serie

so, $\log_2 a_n \to 1$, Therefore $a_n \to 2^1 = \boxed{2}$

§12.2

3, 5, 7 check w/ answers on the back of your text books

1. $3+2+\frac{4}{3}+\frac{8}{9}+\cdots=\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}\cdot 3=\frac{3}{1-\frac{2}{3}}=\boxed{9}$

3. $-2+\frac{5}{2}-\frac{25}{8}+\frac{125}{32}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{5}{4}\right)^{n-1}\cdot 2$

The sequence $\left(\frac{5}{4}\right)^{n-1}$ is divergent, so is $(-1)^n\left(\frac{5}{4}\right)^{n-1}\cdot 2$. Hence this series is $\boxed{\text{divergent}}$.

5. $\sum_{n=1}^{\infty}5\left(\frac{2}{3}\right)^{n-1}=\frac{5}{1-\frac{2}{3}}=\boxed{15}$

7. $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{n-1}}{4^{n-1}\cdot 4}=\sum_{n=1}^{\infty}\frac{1}{4}\left(-\frac{3}{4}\right)^{n-1}=\frac{1/4}{1+\frac{3}{4}}=\boxed{\frac{1}{7}}$

19. $\sum_{n=0}^{\infty}\frac{\pi^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{\pi}{3}\right)^n$ Since $\left|\frac{\pi}{3}\right|>1$, $\left(\frac{\pi}{3}\right)^n$ is divergent, so is $\frac{1}{3}\left(\frac{\pi}{3}\right)^n$

$\Rightarrow$ The series is $\boxed{\text{divergent}}$

21. $\sum_{n=1}^{\infty}\frac{n}{n+5}$ Since $\lim_{n\to\infty}\frac{n}{n+5}=1\neq 0$, the series is $\boxed{\text{divergent}}$ by Divergence Test

23. $\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\sum_{n=2}^{\infty}\frac{2}{(n+1)(n-1)}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$ (Telescoping !!!)

$=\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{5}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)+\cdots=1+\frac{1}{2}=\boxed{\frac{3}{2}}$

25. $\sum_{k=2}^{\infty}\frac{k^2}{k^2-1}=\sum_{k=2}^{\infty}\frac{k^2-1+1}{k^2-1}=\sum_{k=2}^{\infty}1+\frac{1}{k^2-1}=\underbrace{\sum_{k=2}^{\infty}1}_{\text{divergent}}+\sum_{k=2}^{\infty}\frac{1}{k^2-1}$

$\Rightarrow\sum_{k=2}^{\infty}\frac{k^2}{k^2-1}$ is $\boxed{\text{divergent}}$.

27. $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^n=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}\frac{1}{3}$

$=\frac{\frac{1}{2}}{1-\frac{1}{2}}+\frac{\frac{1}{3}}{1-\frac{1}{3}}=\boxed{\frac{3}{2}}$