Math 31B, Homework 4

§8.4, Problems $(1, 3, 5)_a$, $7 - 17$ odd, 18.

October 21, 2005

(1-5) *Write out the partial fraction decomposition. Do not determine the numerical values of the coefficients*

1_a. $\frac{2x}{(x+3)(3x+1)}$

$$= \frac{A}{x+3} + \frac{B}{3x+1}$$

3_a. $\frac{2}{x^2+3x-4}$

$$= \frac{2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

5_a. $\frac{x^4}{x^4-1}$

Trying to solve this directly:

$$= \frac{x^4}{(x^2+1)(x^2-1)} = \frac{x^4}{(x^2+1)(x+1)(x-1)}$$

$$= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

Notice that the system of equations

$$\begin{aligned} x^4 &= (Ax+B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x-1) \\ &= (A+C+D)x^3 + (B-C-D)x^2 + (C+D)x + (-C-D) \end{aligned}$$

has no solution. The left-hand side has higher order than anything on the right-hand side. Partial fractions will not be able to rearrange this expression. To remedy this, we bring down the order of the numerator:

$$\frac{x^4}{x^4-1} = \frac{(x^4-1)+1}{x^4-1} = 1 + \frac{1}{x^4-1}$$

Now this is something that we can break into partial fractions

$$1 + \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

(7-18) Evaluate the integral

7. $\int \frac{x}{x-6}\,dx$

$$= \int \frac{(x-6)+6}{x-6}\,dx = \int (1 + \frac{6}{x-6})\,dx$$

$$= x + 6\ln|x-6| + C$$

9. $\int \frac{x-9}{(x+5)(x-2)}\,dx$

$$= \int \frac{x-9}{(x+5)(x-2)}\,dx = \int (\frac{A}{x+5} + \frac{B}{x-2})\,dx$$

$$= A\ln|x+5| + B\ln|x-2| + C$$

Determining the coefficients:

$$\begin{aligned} x-9 &= A(x-2) + B(x+5) \\ &= (A+B)x + (-2A+5B) \end{aligned}$$

If we equate coefficients, we have the system

$$1 = A + B \qquad\qquad -9 = -2A + 5B$$

which has the solution $A = 2,\ B = -1$. So our final answer is

$$2\ln|x+5| - \ln|x+2| + C = \ln\left|\frac{(x+5)^2}{x+2}\right| + C.$$

11. $\int_2^3 \frac{1}{x^2-1}\,dx$

$$= \int_2^3 \frac{1}{(x-1)(x+1)}\,dx = \int_2^3 (\frac{A}{x+1} + \frac{B}{x-1})\,dx$$

$$= [A\ln|x+1| + B\ln|x-1|]_2^3$$

Determining the coefficients:

$$\begin{aligned} 1 &= A(x-1) + B(x+1) \\ &= (A+B)x + (-A+B) \end{aligned}$$

If we equate coefficients, we have the system

$$0 = A + B \qquad\qquad 1 = -A + B$$

which has the solution $A = -1/2,\ B = 1/2$. So our final answer is

$$\frac{1}{2}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\ln\left(\frac{2+1}{2-1}\cdot\frac{3-1}{3+1}\right) = \frac{1}{2}\ln(3/2).$$

13. $\int \frac{ax}{x^2-bx}\,dx$

Oddly, this simplifies to

$$\int \frac{a}{x-b}\,dx = a\ln|x-b| + C.$$

15. $\int_0^1 \frac{2x+3}{(x+1)^2}\,dx$

Let's clear up the denominator with a u-substitution:

$$u = x+1 \qquad\qquad du = dx$$

Then our integral becomes

$$\int_1^2 \frac{2(u-1)+3}{u^2}\,du = \int_1^2 \frac{2u+1}{u^2}\,du$$

$$= \int_1^2 \left(\frac{2}{u} + \frac{1}{u^2}\right) du = \left[2\ln|u| - \frac{1}{u}\right]_1^2$$

$$= 2\ln(2) + \frac{1}{2}$$

(Notice that I've changed the limits of integration from (0,1) to (1,2))

17. $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dx$

$$= \int_1^2 \left(\frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3}\right) dy$$

$$= [A\ln|y| + B\ln|y+2| + C\ln|y-3|]_1^2$$

Determining the coefficients:

$$\begin{aligned} 4y^2 - 7y - 12 &= A(y+2)(y-3) + B(y)(y-3) + C(y)(y+2) \\ &= (A+B+C)y^2 + (-A-3B+2C)y - 6A \end{aligned}$$

If we equate coefficients, we have the system

$$4 = A+B+C \qquad -7 = -A-3B+2C \qquad -12 = -6A$$

which has the solution $A = 2$, $B = 9/5$, $C = 1/5$. So our final answer is

$$\left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln(2/1) + \frac{9}{5}\ln(4/3) + \frac{1}{5}\ln(1/2) = \frac{9}{5}\ln(8/3)$$

18. $\int \frac{x^2+2x-1}{x^3-x}\,dx$

A little rearrangement turns what would be a system of three equations into a system of two equations (which we have solved already).

$$= \int \frac{(x+1)(x-1)+2x}{x(x+1)(x-1)}\,dx = \int (\frac{1}{x} + \frac{2}{(x+1)(x-1)})\,dx$$

$$= \int (\frac{1}{x} + 2(\frac{A}{x+1} + \frac{B}{x-1}))\,dx = \ln|x| + 2(A\ln|x+1| + B\ln|x-1|).$$

From (11), we know that $A = -1/2$ and $B = 1/2$. Our final solution is then

$$\ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x^2-x}{x+1}\right| + C.$$