31B HW3

6.5 7-10, 13
8.1 1-11 odd 29-34, 45
8.2 1-15 odd 31-45 odd
8.3 1-21 odd

Prepared by Mike Hofman

6.5 7-10, 13

(7) $h(x) = \cos^4 x \sin x$ on $[0,\pi]$

$$h_{ave} = \frac{1}{\pi - 0}\int_0^\pi \cos^4 x \sin x \, dx$$

let $u = \cos x$

$du = -\sin x\,dx$ thus

$$h_{ave} = \frac{1}{\pi}\left(-\frac{\cos^5 x}{5}\Big|_0^\pi\right) = \frac{1}{\pi}\left(-\frac{(-1)^5}{5} - -\frac{1}{5}\right)$$

$$= \boxed{\frac{2}{5\pi}}$$

(8). $h(r) = 3/(1+r)^2$ on $[1,6]$

$$h_{ave} = \frac{1}{6-1}\int_1^6 \frac{3}{(1+r)^2}\,dr = \frac{1}{5}\cdot 3\left(-\frac{1}{(1+r)}\Big|_1^6\right)$$

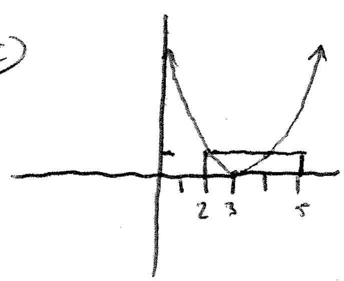
$$= \frac{3}{5}\left(\frac{1}{2} - \frac{1}{7}\right) = \boxed{\frac{3}{14}}$$

(9) $f(x) = (x-3)^2$ $[2,5]$

(a) $f_{ave} = \frac{1}{5-2}\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left(\frac{(x-3)^3}{3}\Big|_2^5\right) = \frac{1}{3}\left(\frac{8}{3} - \frac{(-1)}{3}\right) = \boxed{1}$

(b) $f(c) = 1 \Rightarrow (c-3)^2 = 1$ $c-3 = \pm 1 \Rightarrow c = \pm 1 + 3 = \boxed{2, 4}$

(c)



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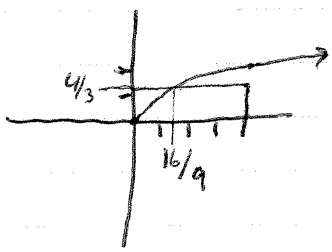
Section 6.5

(10) $f(x) = \sqrt{x}$ on $[0,4]$

(a) $f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\left(\frac{2x^{3/2}}{3}\Big|_0^4\right) = \frac{1}{4}\left(\frac{16}{3} - 0\right) = \boxed{\frac{4}{3}}$

(b) $f(c) = \frac{4}{3} \Rightarrow \sqrt{c} = \frac{4}{3} \Rightarrow$ ~~$c = \frac{9}{16}$~~ $c = \boxed{\frac{16}{9}}$

(c)



(13) f is continuous & $\int_1^3 f(x)\,dx = 8$

if f is never equal to 4 on $[1,3]$ that means that either $f > 4$ on $[1,3]$ or $f < 4$ on $[1,3]$ since f is continuous (Intermediate Value theorem)

If $f(x) > 4$ on $[1,3]$ $\Rightarrow 8 = \int_1^3 f(x)\,dx > \int_1^3 4\,dx = 8$
but $8 > 8$ is a contradiction.

Similarly $f(x) < 4$ on $[1,3]$ $\Rightarrow$ $8 = \int_1^3 f(x)\,dx < \int_1^3 4\,dx = 8$
$8 < 8$ Contradiction

Thus since we can't have $f(x) < 4$ or $f(x) > 4$ on all of $[1,3]$
$\Rightarrow$ there must be at least one point c where $f(c) = 4$.

Section 8.1

8.1 1-11 odd 29-34, 45

Recall our formula $\int u\,dv = uv - \int v\,du$

(1) $\int x \ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\frac{1}{x}dx$

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = x\,dx \quad v = \frac{x^2}{2}$

$= \boxed{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C}$

(3) $\int x \cos 5x\,dx = x\frac{\sin 5x}{5} - \int \frac{\sin 5x}{5}dx$

$u = x \quad du = dx$

$dV = \cos 5x\,dx \quad v = \frac{\sin 5x}{5}$

$= \boxed{\frac{x \sin 5x}{5} + \frac{\cos 5x}{25} + C}$

(5) $\int r e^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr$

$u = r \quad du = dr$

$dv = e^{r/2}dr \quad V = 2e^{r/2}$

$= \boxed{2re^{r/2} - 4e^{r/2} + C}$

(7) $\int x^2 \sin \pi x\,dx = -\frac{x^2\cos\pi x}{\pi} - \int -\frac{\cos \pi x}{\pi}\cdot 2x\,dx$

$u = x^2 \quad du = 2x\,dx$

$dV = \sin\pi x\,dx \quad V = -\frac{\cos\pi x}{\pi}$

$$= -\frac{x^2\cos\pi x}{\pi} + \frac{2}{\pi}\int x\cos\pi x\,dx \quad *$$

integrating by parts again $u = x \quad du = dx$

$dV = \cos\pi x \quad v = \frac{\sin\pi x}{\pi}$

$$= -\frac{x^2\cos\pi x}{\pi} + \frac{2}{\pi}\left(\frac{x\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}dx\right)$$

$$= \boxed{-\frac{x^2\cos\pi x}{\pi} + \frac{2x\sin\pi x}{\pi^2} + \frac{2}{\pi^3}\cos\pi x + C}$$

Section 8.1

(9) $\int \ln(2x+1)\,dx = x\ln(2x+1) - \int x\left(\frac{2}{2x+1}\right)dx$

$u = \ln(2x+1) \quad du = \frac{2}{2x+1}dx$

$dv = dx \quad v = x$

$= x\ln(2x+1) - \int \frac{2x+1-1}{2x+1}dx = x\ln(2x+1) - \int 1 - \frac{1}{2x+1}dx$

$= x\ln(2x+1) - x + \frac{\ln(2x+1)}{2} + C = \boxed{\frac{1}{2}(2x+1)\ln(2x+1) - x + C}$

(11) $\int \tan^{-1}(4t)\,dt = t\tan^{-1}(4t) - \int \frac{4t}{1+(4t)^2}dt =$

$u = \tan^{-1}(4t) \quad du = \frac{4}{1+(4t)^2}dt$

$dV = dt \quad v = t$

$= \boxed{t\tan^{-1}(4t) - \frac{1}{8}\ln(1+(4t)^2) + C}$

(29) $\int \cos(\ln x)\,dx = x\cos(\ln x) - \int x\left(\frac{-\sin(\ln(x))}{x}\right)dx$

$u = \cos(\ln x) \quad du = -\frac{\sin(\ln x)}{x}dx$

$dv = dx \quad v = x$

$= x\cos(\ln x) + \int \sin(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int x\frac{\cos\ln x}{x}dx$

lets integrate by parts again

$u = \sin(\ln x) \quad du = \frac{\cos(\ln x)}{x}dx$

$dV = dx \quad v = x$

thus $\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

~~these two terms are the same~~ Note integrals are the same

$\Rightarrow \quad 2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$

$\Rightarrow \quad \int \cos(\ln x)\,dx = \boxed{\frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C}$

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Section 8.1

(30) $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left(r^2\sqrt{4+r^2}\right|_0^1 - \int_0^1 2r\sqrt{4+r^2}\,dr$

$u = r^2 \quad du = 2r\,dr$
$dv = \frac{r}{\sqrt{4+r^2}}\,dr \quad v = \sqrt{4+r^2}$

$= \left(1^2\sqrt{4+1} - 0\right) - \left(\frac{2(4+r^2)^{3/2}}{3}\right|_0^1$

$= \sqrt{5} - \left(\frac{10\sqrt{5}}{3} - \frac{16}{3}\right) = \boxed{\frac{16}{3} - \frac{7\sqrt{5}}{3}}$

(31) $\int_1^2 x^4(\ln x)^2\,dx = \left((\ln x)^2 \frac{x^5}{5}\right|_1^2 - \int_1^2 \frac{2\ln x}{x}\,\frac{x^5}{5}\,dx$

$u = (\ln x)^2 \quad du = 2\ln x\,(1/x)\,dx$
$dv = x^4\,dx \quad v = x^5/5$

$= \left((\ln 2)^2 \frac{32}{5} - 0\right) - \frac{2}{5}\int_1^2 x^4 \ln x\,dx$

$u = \ln x \quad du = 1/x\,dx$
$dv = x^4\,dx \quad v = x^5/5$

$= \frac{32(\ln 2)^2}{5} - \frac{2}{5}\left(\left(\frac{x^5 \ln x}{5}\right|_1^2 - \int_1^2 \frac{x^4}{5}\,dx\right)$

$= \frac{32(\ln 2)^2}{5} - \frac{2}{5}\left(\left(\frac{32}{5}\ln 2 - 0\right) - \frac{1}{5}\left(\frac{x^5}{5}\right|_1^2\right)$

$= \boxed{\frac{32(\ln 2)^2}{5} - \frac{64}{25}\ln 2 + \frac{62}{125}}$

(32) $\int_0^t e^s \sin(t-s)\,ds = \left(e^s \sin(t-s)\right|_0^t - \int_0^t -e^s\cos(t-s)\,ds$

let $u = \sin(t-s) \quad du = -\cos(t-s)\,ds$
$dv = e^s\,ds \quad v = e^s$

$= \left(e^t \sin 0 - e^0 \sin t\right) + \int_0^t e^s\cos(t-s)\,ds$

do integration by parts again $u = \cos(t-s) \quad du = \sin(t-s)$
$dv = e^s\,ds \quad v = e^s$

$= -\sin t + \left(e^s\cos(t-s)\right|_0^t - \int_0^t e^s \sin(t-s)\,ds$

$\Rightarrow 2\int_0^t e^s\sin(t-s)\,ds = -\sin t + \left(e^t\cos 0 - e^0\cos t\right) \Rightarrow \int_0^t e^s\sin(t-s)\,ds = \frac{-\sin t - \cos t + e^t}{\ldots}$

Section 8.1

(33) $\int \sin\sqrt{x}\,dx \qquad \theta = \sqrt{x} \Rightarrow \theta^2 = x, \quad 2\theta d\theta = dx$

$= \int (\sin\theta)\, 2\theta d\theta = 2\theta(-\cos\theta) - \int -2\cos\theta\, d\theta$

let $u = 2\theta \quad du = 2\,d\theta$

$dv = \sin\theta d\theta \quad v = -\cos\theta$

$= -2\theta\cos\theta + 2\sin\theta + C = \boxed{-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C}$

(34) $\int_1^4 e^{\sqrt{x}}\,dx \qquad \theta = \sqrt{x} \quad \theta^2 = x \quad 2\theta d\theta = dx$

$x = 1..4 \Rightarrow \theta = 1..2$

$= \int_1^2 e^{\theta}\, 2\theta d\theta = (2\theta e^{\theta}|_1^2 - \int_1^2 2e^{\theta} d\theta$

$u = \theta \quad du = 2d\theta$

$dv = e^{\theta}d\theta. \quad v = e^{\theta}$

$= (2e^2 - e) - (2e^{\theta}|_1^2$

$= (2e^2 - e) - (2e^2 - 2e) = \boxed{e}$

(45) $\int (\ln x)^n\,dx = x(\ln x)^n - \int x \frac{n(\ln x)^{n-1}}{x}\,dx$

$u = (\ln x)^n \quad du = \frac{n(\ln x)^{n-1}}{x}\,dx$

$dv = dx \quad v = x$

$= \boxed{x(\ln x)^n - n\int (\ln x)^{n-1}\,dx}$

Section 8.2

8.2 1-15 odd, 31-45 odd

(1) $\int \sin^3 x \cos^2 x \, dx = \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

$= \int \sin x \cos^2 x \, dx - \int \sin x \cos^4 x \, dx$

$u = \cos x \quad du = -\sin x$

$= -\frac{u^3}{3} + \frac{u^5}{5} = \boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$

(3) $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x \, dx = \int_{\pi/2}^{3\pi/4} \sin^5 x (1 - \sin^2 x) \cos x \, dx$

$= \int \sin^5 x \cos x \, dx - \int \sin^7 x \cos x \, dx$

$= \left(\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} \right) \Big|_{\pi/2}^{3\pi/4} = \left(\frac{(\sqrt{2}/2)^6}{6} - \frac{(\sqrt{2}/2)^8}{8} \right) - \left(\frac{1}{6} - \frac{1}{8} \right)$

$= \frac{1}{48} - \frac{1}{128} - \frac{1}{6} + \frac{1}{8} = \boxed{\frac{-11}{384}}$

(5) $\int \cos^5 x \sin^4 x \, dx = \int \cos x (1 - \sin^2 x)^2 \sin^4 x \, dx$

$= \int \cos x \sin^4 x - 2 \int \cos x \sin^6 x \, dx + \int \cos x \sin^8 x \, dx$

$= \boxed{\frac{\sin^5 x}{5} - 2 \frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + C}$

(7) $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} 1 + \cos 2x \, dx = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) \Big|_0^{\pi/2}$

$= \frac{1}{2} \left(\pi/2 + 0 - (0 + 0) \right) = \boxed{\frac{\pi}{4}}$

Section 8.2

(9) $\int_0^{\pi} \sin^4(3t)\,dt = \frac{1}{3}\int_0^{3\pi} \sin^4(u)\,du = \frac{1}{3}\int_0^{3\pi}\left(\frac{1-\cos 2u}{2}\right)^2 du$

let $u = 3t$

$du = 3dt$

$t = 0..\pi$

$\Rightarrow u = 0..3\pi$

$= \frac{1}{12}\int_0^{3\pi} 1 - 2\cos 2u + \cos^2 2u\,du$

$= \frac{1}{12}\int_0^{3\pi} 1 - 2\cos 2u + \frac{1+\cos 4u}{2}\,du$

$= \frac{1}{12}\left(u - \sin 2u + \frac{1}{2}u + \frac{\sin 4u}{8}\right)\Big|_0^{3\pi} = \frac{1}{12}\left[\left(3\pi - 0 + \frac{3\pi}{2} + 0\right) - (0 - 0 + 0 + 0)\right]$

$= \boxed{\frac{9\pi}{24}} = \boxed{\frac{3\pi}{8}}$

(11) $\int (1+\cos\theta)^2\,d\theta = \int 1 + 2\cos\theta + \cos^2\theta\,d\theta = \int 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta$

$= \theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$

$= \boxed{\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4} + C}$

(13) $\int_0^{\pi/4} \sin^4 x\cos^2 x\,dx = \int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)^2\left(\frac{1+\cos 2x}{2}\right)dx$

$= \frac{1}{8}\int_0^{\pi/4}(1 - 2\cos 2x + \cos^2 2x)(1+\cos 2x)\,dx$

$= \frac{1}{8}\int_0^{\pi/4} 1 - \cos 2x - \cos^2 2x + \cos^3 2x\,dx$

$= \frac{1}{8}\left[\int_0^{\pi/4} 1 - \cos 2x\,dx - \int_0^{\pi/4}\frac{1+\cos 4x}{2}\,dx + \int_0^{\pi/4}\cos 2x(1-\sin^2 2x)\,dx\right]$

$= \frac{1}{8}\left[x - \frac{\sin 2x}{2} - \frac{1}{2}x - \frac{\sin 4x}{8} + \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}\right|_0^{\pi/4}$

$= \frac{1}{8}\left[\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\pi}{8} - 0 + \frac{1}{2} - \frac{1}{6}\right) - (0)\right]$

$= \boxed{\frac{\pi}{64} - \frac{1}{48}} = \boxed{\frac{3\pi-4}{192}}$

Section 8.2

(15) $\int \sin^3 x \sqrt{\cos x}\,dx = \int \sin x\,(1-\cos^2 x)\sqrt{\cos x}\,dx$

$= \int \sin x\,(\cos x)^{1/2} - \sin x(\cos x)^{5/2}\,dx$

$= \boxed{-\frac{2(\cos x)^{3/2}}{3} + \frac{2(\cos x)^{7/2}}{7} + C}$

(37) $\int \tan^5 x\,dx = \int \tan^3 x(\sec^2 x - 1)\,dx$

$= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$

$= \frac{\tan^4 x}{4} - \int \tan x(\sec^2 x - 1)\,dx$

$= \boxed{\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + C}$

note $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} = \frac{(\sec^2 x - 1)^2}{4} - \frac{\tan^2 x}{2} = \frac{\sec^4 x - 2\sec^2 x + 1}{4} - \frac{\tan^2 x}{2}$

$= \frac{\sec^4 x}{4} + \frac{1}{4} - \frac{\sec^2 x}{2} - \frac{\tan^2 x}{2} = \frac{\sec^4 x}{4} + \frac{1}{4} - \frac{1+\tan^2 x}{2} - \frac{\tan^2 x}{2}$

$= \frac{\sec^4 x}{4} - \tan^2 x - \frac{1}{4}$

thus $\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + C = \boxed{\frac{\sec^4 x}{4} - \tan^2 x + \ln|\sec x| + C'}$

where $C' = C - \frac{1}{4}$ So our answer agrees with the Book answer.

(33) $\int \frac{\tan^3\theta}{\cos^4\theta}\,d\theta = \int \tan^3\theta \sec^4\theta\,d\theta = \int \tan^3\theta\,(1+\tan^2\theta)\sec^2\theta\,d\theta$

$= \int \tan^3\theta \sec^2\theta\,d\theta + \int \tan^5\theta \sec^2\theta\,d\theta$

$= \boxed{\frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} + C}$

(35) $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} \csc^2 x - 1\,dx = \left(-\cot x - x\right|_{\pi/6}^{\pi/2}$

$= (0 - \pi/2) - (-\sqrt{3} - \pi/6) = \boxed{\sqrt{3} - \frac{\pi}{3}}$

(37) $\int \cot^3\alpha \csc^3\alpha\,d\alpha = \int \frac{\cos^3\alpha}{\sin^3\alpha} \cdot \frac{1}{\sin^3\alpha}\,d\alpha = \int \frac{\cos\alpha(1-\sin^2\alpha)}{\sin^6\alpha}\,d\alpha$

$= \int \cos\alpha \sin^{-6}\alpha\,d\alpha - \int \cos\alpha \sin^{-4}\alpha\,d\alpha = \frac{\sin^{-5}\alpha}{-5} - \frac{\sin^{-3}\alpha}{-3} + C$

$= \boxed{-\frac{1}{5}\csc^5\alpha + \frac{1}{3}\csc^3\alpha + C}$

(39) $\int \csc x\,dx = \int \csc x \frac{(\csc x - \cot x)}{(\csc x - \cot x)}\,dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\,dx$

let $u = \csc x - \cot x$
$du = (-\csc x \cot x + \csc^2 x)\,dx$

$= \int \frac{1}{u}\,du = \ln|u| + C$

$= \boxed{\ln|\csc x - \cot x| + C}$

(41) $\int \sin 5x \sin 2x\,dx = \int \frac{1}{2}(\cos 3x - \cos 7x)\,dx$

$= \boxed{\frac{1}{2}\left(\frac{\sin 3x}{3} - \frac{\sin 7x}{7}\right) + C}$

(43) $\int \cos 7\theta \cos 5\theta\,d\theta = \int \frac{1}{2}(\cos 2\theta + \cos 12\theta)\,d\theta$

$= \boxed{\frac{1}{2}\left(\frac{\sin 2\theta}{2} + \frac{\sin 12\theta}{12}\right) + C}$

(45) $\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx = \int \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1/\cot x}\,dx = \int \cos^2 x - \sin^2 x\,dx$

$= \int \cos 2x\,dx = \boxed{\frac{\sin 2x}{2} + C}$

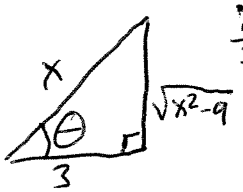
8.3 1-21 odd

(1)

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-9}}\,3\sec\theta\tan\theta\,d\theta$$

$x = 3\sec\theta$

$\frac{x}{3} = \sec\theta = \frac{hyp}{adj}$



$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\cdot 3\tan\theta}$$

$$= \int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta + C$$

$$= \frac{1}{9}\sin\theta\,d\theta = \frac{1}{9}\sqrt{1-\cos^2\theta} + C$$

$$= \frac{1}{9}\sqrt{1-\frac{1}{\sec^2\theta}} + C = \frac{1}{9}\sqrt{1-\frac{1}{\sec^2(\sec^{-1}(\frac{x}{3}))}} + C$$

$$= \frac{1}{9}\sqrt{1-\frac{1}{(x/3)^2}} = \boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$

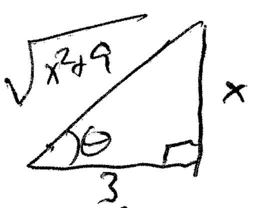
(3)

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(3\tan\theta)^3}{\sqrt{(3\tan\theta)^2+9}}(3\sec^2\theta\,d\theta)$$

$x = 3\tan\theta$

$x/3 = \tan\theta = \frac{opp}{adj}$

$$= \int \frac{27\tan^3\theta\sec^2\theta\,d\theta}{3\sec\theta}$$



$dx = 3\sec^2\theta\,d\theta$

$$= 9\int \tan^3\theta\sec\theta\,d\theta$$

$$= 9\int \tan\theta(\sec^2\theta-1)\sec\theta\,d\theta$$

$$= 9\left[\int \sec^2\theta(\tan\theta\sec\theta)\,d\theta - \int \sec\theta\tan\theta\,d\theta\right.$$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$$\boxed{9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C}$$

$$= 9\left[\frac{(x^2+9)^{3/2}}{81} - \frac{(x^2+9)^{1/2}}{3}\right) + C$$

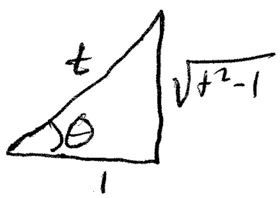
$$= \frac{(x^2+9)^{3/2}}{9} - 3(x^2+9)^{1/2} + C$$

(5)

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta\,d\theta)$$

$t = \sec\theta = \frac{hyp}{adj}$

$dt = \sec\theta\tan\theta\,d\theta$



$$= \int \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{\sin 2\theta}{4}$$

$\sin\theta = \frac{\sqrt{t^2-1}}{t}$

$\cos\theta = \frac{1}{t}$

$$= \frac{1}{2}\theta + \frac{2\sin\theta\cos\theta}{4}$$

$$= \left(\frac{1}{2}\sec^{-1}t + \frac{1}{2}\frac{\sqrt{t^2-1}}{t}\cdot\frac{1}{t}\right)\Bigg|_{\sqrt{2}}^{2} = \frac{1}{2}\left(\sec^{-1}t + \frac{\sqrt{t^2-1}}{t^2}\right)\Bigg|_{\sqrt{2}}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)-\left(\frac{\pi}{4}+\frac{\sqrt{1}}{2}\right)\right]$$

$$= \boxed{\frac{\pi}{24} - \frac{1}{4} + \frac{\sqrt{3}}{8}}$$

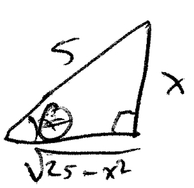
(7)

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{(5\sin\theta)^2\sqrt{25-25\sin^2\theta}}(5\cos\theta\,d\theta)$$

$x = 5\sin\theta$

$\frac{x}{5} = \sin\theta = \frac{opp}{hyp}$

$$= \int \frac{5\cos\theta\,d\theta}{25\sin^2\theta\cdot 5\cos\theta}$$



$dx = 5\cos\theta\,d\theta$

$$= \frac{1}{25}\int \csc^2\theta\,d\theta =$$

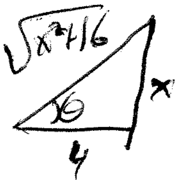
$$= -\frac{1}{25}\cot\theta + C$$

$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{25-x^2}}{x}$

$$\boxed{= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C}$$

(9) $\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta \, d\theta}{\sqrt{16\tan^2\theta+16}} = \int \frac{4\sec^2\theta}{4\sec\theta} d\theta$

$x = 4\tan\theta$



$\frac{x}{4} = \tan\theta = \frac{opp}{adj}$

$dx = 4\sec^2\theta \, d\theta$

$= \int \sec\theta \, d\theta$

$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)} d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\tan\theta+\sec\theta} d\theta$

$\tan\theta = \frac{x}{4}$

$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+16}}{4}$

$= \ln|\tan\theta + \sec\theta|$

$= \boxed{\ln\left|\frac{x}{4} + \frac{\sqrt{x^2+16}}{4}\right| + C}$

(11) $\int \sqrt{1-4x^2}\, dx$

$= \int \sqrt{1-\sin^2\theta}\ \frac{1}{2}\cos\theta \, d\theta = \int \frac{1}{2}\cos^2\theta \, d\theta$

$x = \frac{1}{2}\sin\theta$

$2x = \sin\theta = \frac{opp}{hyp}$

$dx = \frac{1}{2}\cos\theta \, d\theta$

$= \frac{1}{2}\int \frac{1+\cos 2\theta}{2} d\theta$

$= \frac{1}{4}\theta + \frac{\sin 2\theta}{8} + C$

$= \frac{1}{4}\theta + \frac{2\sin\theta\cos\theta}{8} + C$

$\sin\theta = 2x$

$\cos\theta = \sqrt{1-4x}$

$\theta = \sin^{-1}(2x)$

$= \boxed{\frac{1}{4}(\theta + \sin\theta\cos\theta) + C}$

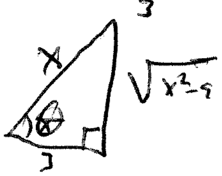
$\boxed{\frac{1}{4}\left(\sin^{-1}2x + 2x\sqrt{1-4x}\right) + C}$

(13) $\int \frac{\sqrt{x^2-9}}{x^3} dx$

$= \int \frac{\sqrt{9\sec^2\theta-9}}{(3\sec\theta)^3} 3\sec\theta\tan\theta \, d\theta = \int \frac{(3\tan\theta)(3\sec\theta\tan\theta)}{27\sec^3\theta} d\theta$

$x = 3\sec\theta$

$\frac{x}{3} = \sec\theta = \frac{hyp}{adj}$



$dx = 3\sec\theta\tan\theta$

$= \int \frac{\tan^2\theta}{3\sec^2\theta} d\theta = \frac{1}{3}\int \sin^2\theta \, d\theta$

$= \frac{1}{3}\int \frac{1}{2} - \frac{\cos 2\theta}{2} d\theta = \frac{1}{6}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$

$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$\cos\theta = \frac{3}{x}$

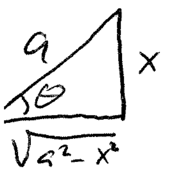
$\theta = \sec^{-1}\left(\frac{x}{3}\right)$

$= \frac{1}{6}\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C = \boxed{\frac{1}{6}\left(\sec^{-1}\left(\frac{x}{3}\right) - \frac{3\sqrt{x^2-9}}{x^2}\right) + C}$

5) $\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx = \int \frac{(a\sin\theta)^2}{\left(\sqrt{a^2-a^2\sin^2\theta}\right)^3}\, a\cos\theta\, d\theta$

$x = a\sin\theta$

$x/a = \sin\theta = \frac{opp}{hyp}$



$dx = a\cos\theta\, d\theta$

$\theta = \sin^{-1}\frac{x}{a}$

$\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$

$= \int \frac{a^2\sin^2\theta\, a\cos\theta}{a^3\cos^3\theta}\,d\theta$

$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

$= \int \frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta = \int \frac{1}{\cos^2\theta}\,d\theta - \int d\theta$

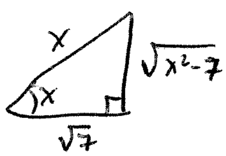
$= \int \sec^2\theta - 1\, d\theta = \tan\theta - \theta + C$

$= \boxed{\frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C}$

17) $\int \frac{x}{\sqrt{x^2-7}}\,dx = \int \frac{\sqrt{7}\sec\theta}{\sqrt{7\sec^2\theta - 7}}\,\sqrt{7}\sec\theta\tan\theta\, d\theta$

$x = \sqrt{7}\sec\theta$

$\frac{x}{\sqrt{7}} = \sec\theta = \frac{hyp}{adj}$



$dx = \sqrt{7}\sec\theta\tan\theta\, d\theta$

$\tan\theta = \frac{\sqrt{x^2-7}}{\sqrt{7}}$

$= \int \frac{\sqrt{7}\sec\theta\,\sqrt{7}\sec\theta\tan\theta}{\sqrt{7}\tan\theta}\,d\theta$

$= \sqrt{7}\int \sec^2\theta\, d\theta$

$= \sqrt{7}\tan\theta + C$

$= \sqrt{7}\,\frac{\sqrt{x^2-7}}{\sqrt{7}} + C$

$= \boxed{\sqrt{x^2-7} + C}$

(19) $\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sqrt{u}}{x}\frac{du}{2x} = \frac{1}{2}\int \frac{\sqrt{u}}{u-1}\,du$

$u = 1+x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \frac{\sqrt{u}-1+1}{u-1}\,du = \frac{1}{2}\int \frac{\sqrt{u}-1}{u-1} + \frac{1}{u-1}\,du$

$= \frac{1}{2}\int \frac{(\sqrt{u}-1)}{(\sqrt{u}-1)(\sqrt{u}+1)}\,du + \frac{1}{2}\int \frac{1}{u-1}\,du = \frac{1}{2}\int \frac{du}{\sqrt{u}+1} + \frac{1}{2}\ln|u-1| + c$

let $v = \sqrt{u}+1$

$dv = \frac{1}{2\sqrt{u}}\,du$

$= \frac{1}{2}\int \frac{2\sqrt{u}}{v}\,dv + \frac{1}{2}\ln|u-1| + c$

$= \int \frac{v-1}{v}\,dv + \frac{1}{2}\ln|u-1| + C$

$= \int 1 - \frac{1}{v}\,dv + \frac{1}{2}\ln|u-1| + C$

$= v - \ln|v| + \frac{1}{2}\ln|u-1| + C$

$= \sqrt{u}+1 - \ln|\sqrt{u}+1| + \frac{1}{2}\ln|u-1| + C$

$= \boxed{\sqrt{x^2+1} + 1 - \ln|\sqrt{x^2+1}+1| + \frac{1}{2}\ln|x^2| + C}$

(21) $\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 x^3\sqrt{u}\,\frac{du}{-18x} = -\frac{1}{18}\int_4^0 \left(\frac{4-u}{9}\right)\sqrt{u}\,du$

$u = 4-9x^2 \quad x^2 = \frac{4-u}{9}$

$du = -18x\,dx$

$x = 0..2/3$

$u = 4..0$

$= \frac{1}{18\cdot 9}\int_0^4 4\sqrt{u} - u^{3/2}\,du$

$= \frac{1}{162}\left(4u^{3/2}\left(\frac{2}{3}\right) - u^{5/2}\left(\frac{2}{5}\right)\right|_0^4$

$= \frac{1}{162}\left[\left(\frac{8}{3}\cdot 8 - \frac{2}{5}\cdot 32\right) - (0-0)\right] = \frac{1}{162}\left[\frac{64}{3} - \frac{64}{5}\right]$

$= \frac{128}{2430} = \boxed{\frac{64}{1215}}$