§7-5

(1) ⓐ Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ //

ⓑ Since $\cos(\pi) = -1 \Rightarrow \cos^{-1}(1) = \pi$. //

(2) ⓐ Since $\tan\left(-\frac{\pi}{4}\right) = -1 \Rightarrow \tan^{-1}(-1) = -\frac{\pi}{4}$ //

ⓑ $2 = \csc\left(\csc^{-1} 2\right) = \dfrac{1}{\sin\left(\csc^{-1} 2\right)}$

$\sin\left(\csc^{-1} 2\right) = 1/2 \Rightarrow \csc^{-1} 2 = \frac{\pi}{6}$ //

(3) ⓐ $\tan\left(\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ //

ⓑ $\sin\left[\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)\right] = \frac{-1}{\sqrt{2}} \Rightarrow \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ //

(4) ⓐ $\sqrt{2} = \sec\left(\sec^{-1}\sqrt{2}\right) = \dfrac{1}{\cos\left(\sec^{-1}\sqrt{2}\right)}$

$\Rightarrow \cos\left(\sec^{-1}\sqrt{2}\right) = \frac{1}{\sqrt{2}} \Rightarrow \sec^{-1}\sqrt{2} = \frac{\pi}{4}$ //

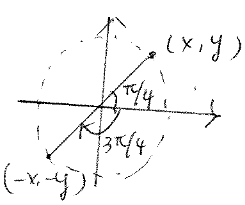
(5) ⓐ $\cos^{-1}(\cos 2\pi) = \cos^{-1}(\cos 0) = 0$ //

↑ Replace 2π with value in $[0, \pi]$ that gives the same cos. value.

ⓑ $\tan(\tan^{-1} 5) = 5$ // ($\tan^{-1}$ has domain on all $\mathbb{R}$)

(6) ⓐ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ //

↑ Replace $\frac{3\pi}{4}$ with value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that gives the same tan. value



$\frac{-x}{-y} = \frac{x}{y}$

$\therefore \tan\frac{3\pi}{4} = \tan\frac{\pi}{4}$

ⓑ $\sin^{-1}\left(\frac{1}{2}\right) = \pi \Rightarrow \cos\left(\sin^{-1}\frac{1}{2}\right)$

$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ //

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(19) let $y = \cot^{-1} x$, $\frac{dy}{dx} = ?$

Since $\cot y = \cot(\cot^{-1} x) = x$

Differentiating w.r.t. x on both sides, & using chain rule, we get

$$-\csc^2 y \cdot \frac{dy}{dx} = 1, \quad \text{OR} \quad \frac{dy}{dx} = -\frac{1}{\csc^2 y} \quad \cdots (*)$$

Recall identity $1 + \cot^2\theta = \csc^2\theta$

$*$ becomes, $\frac{dy}{dx} = -\frac{1}{1 + \cot^2 y}$

$$= -\frac{1}{1 + [\cot(\cot^{-1} x)]^2} = -\frac{1}{1 + x^2}$$

▨

(20) Following similar steps to (19),

let $y = \sec^{-1} x$, & $\sec y = x$

$\therefore (\sec y \tan y) \cdot \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\underbrace{\sec y}_{x} \tan y} \quad \cdots *$$

Recall identity $\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \tan\theta = \pm\sqrt{\sec^2\theta - 1}$

To choose the sign for $\tan y$ in $*$, we note that the range of $\sec^{-1} x$ "y" is $[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, and tan. has positive values there.

$\therefore \tan y = \sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$, putting into $*$ & the equality is proved.

▨

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(22) $y = \sqrt{\tan^{-1} x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{d}{dx}(\tan^{-1} x) \quad ; \quad (\text{chain rule})$$

$$= \frac{1}{2\sqrt{\tan^{-1} x}\,(1+x^2)} \quad //$$

(23) $y = \tan^{-1}(\sqrt{x})$

$$\frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}\sqrt{x})$$

$$= \frac{1}{(1+x)\cdot 2\sqrt{x}} \quad //$$

(24) $h(x) = \sqrt{1-x^2}\,\sin^{-1} x$

$$h'(x) = \frac{d}{dx}\sqrt{1-x^2}\cdot \sin^{-1} x + \frac{d}{dx}(\sin^{-1} x)\cdot\sqrt{1-x^2} \quad ; \quad (\text{product rule}$$

$$= \frac{-\cancel{2}x\cdot\sin^{-1}x}{\cancel{2}\sqrt{1-x^2}} + \frac{1}{\cancel{\sqrt{1-x^2}}}\cdot\cancel{\sqrt{1-x^2}}$$

$$= \frac{\sqrt{1-x^2} - x\sin^{-1}x}{\sqrt{1-x^2}} \quad //$$

(25) $y = \sin^{-1}(2x+1)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot (2x+1)'$$

$$= \frac{2}{\sqrt{1-(2x+1)^2}} \quad //$$

(26) $f(x) = x \cdot \ln(\tan^{-1}x)$

$$f'(x) = x' \cdot \ln(\tan^{-1}x) + x \cdot [\ln(\tan^{-1}x)]'$$

$$= \ln(\tan^{-1}x) + x \frac{(\tan^{-1}x)'}{\tan^{-1}x}$$

$$= \ln(\tan^{-1}x) + \frac{x}{(1+x^2)\tan^{-1}x} \quad //$$

(59) $$\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\sin^{-1}t \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= 6\left(\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}\right)$$

$$= 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \pi \quad //$$

(60) $$\int_0^1 \frac{4}{t^2+1}\,dt = 4\tan^{-1}t \Big|_0^1$$

$$= 4(\tan^{-1}1 - \tan^{-1}0)$$

$$= 4\left(\frac{\pi}{4} - 0\right) = \pi \quad //$$

(61). Let $u = 4x \Rightarrow u^2 = 16x^2$ & $dx = \frac{1}{4}du$

& $\begin{cases} x = 0 \Rightarrow u = 0 \\ x = \frac{\sqrt{3}}{4} \Rightarrow u = \sqrt{3} \end{cases}$

$$\therefore \int_0^{\frac{\sqrt{3}}{4}} \frac{dx}{1+16x^2} = \int_0^{\sqrt{3}} \frac{\frac{1}{4}}{1+u^2}\,du$$

$$= \frac{1}{4}(\tan^{-1}u)\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{4}(\tan^{-1}\sqrt{3} - \tan^{-1}0)$$

$$= \frac{1}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi}{12} \quad //$$

162) let $u = 2t$; $u^2 = 4t^2$ & $dt = \frac{1}{2}du$

$$\therefore \int \frac{dt}{\sqrt{1-4t^2}} = \int \frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u$$

$$= \frac{1}{2}\sin^{-1}(2t) \quad //$$

163) let $u = \sin^{-1}x \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ OR $dx = \sqrt{1-x^2}\,du$

$$\& \begin{cases} x = 0 \Rightarrow u = 0 \\ x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6} \end{cases}$$

$$\therefore \int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{u}{\cancel{\sqrt{1-x^2}}} \cancel{\sqrt{1-x^2}}\,du$$

$$= \frac{u^2}{2}\Big|_0^{\frac{\pi}{6}} = \frac{\pi^2}{72} \quad //$$

164) let $u = \cos x \Rightarrow dx = \frac{-du}{\sin x}$

$$\& \begin{cases} x = 0 \Rightarrow u = 1 \\ x = \frac{\pi}{2} \Rightarrow u = 0 \end{cases}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = \int_1^0 \frac{\cancel{\sin x}}{1+u^2}\frac{du}{-\cancel{\sin x}}$$

$$= \int_0^1 \frac{1}{1+u^2}\,du.$$

the "−" in integrand exchanges the upper & lower limit

$$= \tan^{-1}u\Big|_0^1 = \frac{\pi}{4} \quad //$$

(11)

$$\text{LHS} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \frac{e^x e^y - e^{-x} e^{-y}}{2}$$

$$\text{RHS} = \frac{e^x - e^{-x}}{2}\,\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\,\frac{e^y - e^{-y}}{2}$$

check that they are equal (high school math)

(13)

$$\coth^2 x - 1$$

$$= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1$$

$$= \frac{e^{2x} + e^{-2x} + 2}{e^{2x}\ e^{-2x} - 2} - 1$$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2}{e^{2x} + e^{-2x} - 2}$$

$$= \left(\frac{2}{e^x - e^{-x}}\right)^2 = \operatorname{csch}^2 x \qquad \boxtimes$$

(15)

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}$$

$$= 2 \cdot \frac{e^x + e^{-x}}{2}\,\frac{e^x - e^{-x}}{2} = 2 \sinh x \cosh x \qquad \boxtimes$$

(17)

$$\tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}$$

$$= \frac{x - 1/e^{\ln x}}{x + 1/e^{\ln x}}$$

$$= \frac{x - 1/x}{x + 1/x} = \frac{x^2 - 1}{x^2 + 1} \qquad \boxtimes$$

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(5) indet. form : $0/0$.

$\therefore \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{2x}{1} = -2$ // (L'Hospital)

OR, elementary method.

$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = -2$ //

(7) Elementary method:

$\lim_{x\to 1} \frac{x^9-1}{x^5-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(1+x+x^2+\cdots+x^8)}{\cancel{(x-1)}(1+x+x^2+x^3+x^4)} = \frac{9}{5}$ //

OR, L'Hospital:

$\lim_{x\to 1} \frac{x^9-1}{x^5-1} = \lim_{x\to 1} \frac{9x^8}{5x^4} = \frac{9}{5}$ //

(9) L'Hospital:

$\lim_{x\to (\frac{\pi}{2})^+} \frac{\cos x}{1-\sin x} = \lim_{x\to (\frac{\pi}{2})^+} \frac{-\sin x}{\cos x} = \lim_{x\to (\frac{\pi}{2})^+} -\tan x = -\infty$ //

(11) $\lim_{t\to 0} \frac{e^t-1}{t^3} = \lim_{t\to 0} \frac{e^t}{3t^2} = \infty$ // (always positive as $t\to 0$ from left & right)

(15) $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$ //

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(16) $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$ // (L'Hospital)

(17) $\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$ //

as $x \to 0^+$, $\ln x \to -\infty$; $x \to 0$

positive, very small,
∴ can only enlarge the function without changing the sign.

(18) $\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = 0$ //

(L'Hospital)

(19) $\lim_{t\to 0} \frac{5^t - 3^t}{t} = \lim_{t\to 0} \frac{5^t \ln 5 - 3^t \ln 3}{1} = \ln 5 - \ln 3 = \ln\left(\frac{5}{3}\right)$ //

(L'Hospital)

(20) $\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$ //

(L'Hospital)

(33) $\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} = \lim_{x\to 1} \frac{-1 + 1/x}{-\pi \sin \pi x}$ (L'Hospital)

$= \lim_{x\to 1} \frac{-1/x^2}{-\pi^2 \cos \pi x} = -\frac{1}{\pi^2}$ //

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(34) Elementary Method:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}\,/x}{\sqrt{2x^2+1}\,/x} = \lim_{x\to\infty} \frac{\sqrt{1+\underbrace{2/x^2}_{\xrightarrow{x\to\infty} 0}}}{\sqrt{2+\underbrace{1/x^2}_{\xrightarrow{x\to\infty} 0}}} = 1/\sqrt{2} \;//$$

L'Hospital Not recommended here.

(35) use L'Hospital

$$\lim_{x\to1} \frac{x^a - ax + a - 1}{(x-1)^2} = \lim_{x\to1} \frac{ax^{a-1} - a}{2(x-1)}$$

$$= \lim_{x\to1} \frac{a(a-1)}{2} = \frac{a(a-1)}{2} \;//$$

(36) $$\lim_{x\to0} \frac{1-e^{-2x}}{\sec x} = \frac{0}{1} = 0 \;//$$

(No trick needed)

(37) $$\lim_{x\to0^+} \sqrt{x}\ln x \overset{(\text{L'Hospital})}{=} \lim_{x\to0^+} \frac{\ln x}{x^{-\frac{1}{2}}}$$

$$= \lim_{x\to0^+} \frac{(1/x)}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

$$= \lim_{x\to0^+} -2\sqrt{x} = 0 \;//$$

(38) $$\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{\text{L'Hospital}}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

$$= \lim_{x\to-\infty} \frac{2}{\underbrace{e^{-x}}_{\to\infty}} = 0 \;//$$

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(41)
$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3}{2xe^{x^2}} = 0 \;//$$

(42)
$$\lim_{x\to\frac{\pi}{4}} (1-\tan x)\sec x$$

(No trick needed)

$$= \left(1-\tan\frac{\pi}{4}\right)\frac{1}{\cos\frac{\pi}{4}} = 0 \;//$$

(43)
$$\lim_{x\to 1^+} \ln x \cdot \tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^+} \frac{\ln x}{\cot\left(\frac{\pi x}{2}\right)}$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to 1^+} \frac{1/x}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)}$$

$$= \lim_{x\to 1^+} \frac{2\sin^2\left(\frac{\pi x}{2}\right)}{-\pi x} = -\frac{2}{\pi} \;//$$

(44)
$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \cot\left(\frac{1}{x}\right) \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{}{-\frac{1}{x^2}\csc^2\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} -x^2\sin^2\left(\frac{1}{x}\right)$$

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(44) $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$

$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{-\cancel{\frac{1}{x^2}}\sec^2\left(\frac{1}{x}\right)}{-\cancel{\frac{1}{x^2}}} = 1$ //

*

For 53 ~ 56, we use the following methods:

Note: $\lim_{x\to a} (\ln y(x)) = \ln\left[\lim_{x\to a} y(x)\right]$

$\therefore$ if $\lim_{x\to a} (\ln y(x)) = c \Rightarrow \lim_{x\to a} y(x) = e^c$

For the problems followed, $\lim_{x\to a}[\ln y(x)]$ is much easier to find than $\lim_{x\to a} y(x)$, $\therefore$ this method is applicable. *

(53) let $y = (1-2x)^{\frac{1}{x}}$; $\ln y = \frac{1}{x}\ln(1-2x)$

$\lim_{x\to 0}\ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x} \overset{\text{L'Hospital}}{=} \lim_{x\to 0} \frac{-2}{1-2x} = -2$

$\Rightarrow \lim_{x\to 0} y = e^{-2}$ //

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(54) There is an elementary method for this, if we note that

$$e = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t} ;$$

$$\Rightarrow e^{ab} = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\right]^{ab}$$

$$= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{abt}$$

let $x = at$ then as $t \to \infty$, $x\to\infty$

$$\stackrel{\downarrow}{=} \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} \quad //$$

To use L'Hospital:

let $y = \left(1+\frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx\ln\left(1+\frac{a}{x}\right)$

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}$$

$$= \lim_{x\to\infty}\frac{\frac{-1}{1+\frac{a}{x}}\cdot\frac{a}{\cancel{x^2}}}{-\frac{1}{bx^2}}$$

$$= \lim_{x\to\infty}\frac{ab}{1+\frac{a}{x}} = ab$$

$$\Rightarrow \lim_{x\to\infty} y = e^{ab} \quad //.$$

~~(55) let~~ $\cancel{y = \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^{x}}$

$\cancel{\ln y =}$

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(55) let $y = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$

$$\ln y = x \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)$$

$$\lim_{x \to \infty} (\ln y) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{\frac{1}{x}}$$

L'Hospital

$$= \lim_{x \to \infty} \frac{\frac{x^2\left(-\frac{3}{x^2} - \frac{10}{x^3}\right)}{x^2 + 3x + 5}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{x^4\left(\frac{3}{x^2} + \frac{10}{x^3}\right)}{x^2 + 3x + 5}$$

$$= \lim_{x \to \infty} \frac{3x^2 + 10x}{x^2 + 3x + 5} = 3$$

$$\therefore \lim_{x \to \infty} y = e^3$$

//

(56) let $y = x^{\frac{\ln 2}{1 + \ln x}}$

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x$$

$$\lim_{x \to \infty} (\ln y) = \lim_{x \to \infty} \frac{\ln 2 \cdot \ln x}{1 + \ln x}$$

L'Hospital

$$= \lim_{x \to \infty} \frac{\ln 2 \cancel{(1/x)}}{\cancel{(1/x)}} = \ln 2$$

$$\Rightarrow \lim_{x \to \infty} y = e^2$$

//