MATH 31B SECTIONS 1 AND 3
PRACTICE FINAL EXAM.

Please note: Show your work. Except on multiple-choice problems, correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted.

#1	#2	#3	#4	#5	#6	
#7	#8	#9	#10	#11	#12	Total

SID:____________ TA: ____________ Section(circle): Tuesday Thursday

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Problem 1. Let $f(x) = \sum_{n=0}^{\infty} \frac{3n}{2n+5} x^n$. Find the values of $f(0), f'(0)$ and $f''(0)$.

Recall that for Maclaurin series,

$f(x) = \sum_{n=0}^{\infty} a_n x^n$, we have

$a_n = \frac{f^{(n)}(0)}{n!}$ OR. $f^{(n)}(0) = n!\, a_n$

Here, $a_n = \frac{3n}{2n+5}$

$\therefore \quad f(0) = a_0 = 0$

$f'(0) = a_1 = \frac{3}{2+5} = \frac{3}{7}$

& $f''(0) = 2a_2 = 2 \cdot \frac{6}{9} = \frac{4}{3}$

Problem 2. Let

$$f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x < 0. \end{cases}$$

Show that the function f is one-to-one on $(-\infty, \infty)$ and find its inverse function.

let $x \neq y$, we want to show that $f(x) \neq f(y)$

~~case I: $x \cdot y < 0$ (opposite signs)~~
~~then $f(x)f(y)$ is a product of two numbers of opposite signs, $\therefore$ $f(x)$ and $f(y)$ have opposite signs, & $\therefore$ not equal.~~

~~case II $x \cdot y \geq 0$ (same sign)~~
~~then it follows from the fact that $\sqrt{x}$ $(-\sqrt{-x})$ is a strictly increasing function on all $\mathbb{R}$.~~

Just observe that $f(x)$ is a strictly increasing function on $\mathbb{R}$. that is

$$x < y \Rightarrow f(x) < f(y)$$

and therefore, if $x \neq y$, then either $x < y$ OR $x > y$. then, either $f(x) < f(y)$; OR $f(x) > f(y)$ $\Rightarrow$ $f(x) \neq f(y)$ $\Rightarrow$ one-to-one

to find inverse, we solve x in terms of y

$$y = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases} \Rightarrow \begin{cases} x = y^2, & x \geq 0 \\ x = -y^2, & x < 0 \end{cases} \Rightarrow \boxed{y = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}}$$

inverse

Problem 3. Let $f(x) = \tan^{-1} x$.

(a)Find a power series representation for f around 0. (*Hint:* represent $\tan^{-1} x$ as an integral).

(b) Show the series conveges to $f(x)$ if $-1 < x < 1$.

ⓐ First, we assume that we are safely inside the interval of convergence.

(& ∴ term by term integration is permitted)

Notice that

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\, dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (t^2)^n \, dt \qquad , \text{ (provided } |t|^2 < 1 \text{)}$$

$$= \sum_{n=0}^{\infty} \int_0^x t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

ⓑ If $-1 < x < 1$, $\Rightarrow$ $|x| < 1$

then by ratio test,

$$\rho = \lim_{n \to \infty} \frac{\frac{1}{2n+3}}{\frac{1}{2n+1}} |x| = |x| < 1 \text{ , converges.}$$

and it also ensures that $t^2 < 1$, & so that the geometric series representation is valid.

Problem 4. Is the improper integral $\int_0^\infty x^2 e^{-x} dx$ convergent or divergent? Explain.

Define sequence $\{S_N\}$ by

$$S_N = \int_0^N x^2 e^{-x} dx$$

then $\int_0^\infty x^2 e^{-x} dx = \lim_{N\to\infty} S_N$

S_N is monotonically increasing,

since $S_{N+1} - S_N = \int_N^{N+1} x^2 e^{-x} dx > 0 \quad (x^2 e^{-x} > 0)$

it is also bounded above:

$$\int_0^N x^2 e^{-x} dx \le \int_0^\infty x^2 e^{-x} dx$$

Note that since $\lim_{x\to\infty} x^4 e^{-x} = 0$ (By L'Hospital

there exists M s.t. $x^4 e^{-x} \le 1$ for all $x \ge M$

i.e. $e^{-x} \le \frac{1}{x^4}$ for all $x \ge M$, OR.

$x^2 e^{-x} \le \frac{1}{x^2}$ " " " ".

$$\therefore \int_0^\infty x^2 e^{-x} dx = \int_0^M x^2 e^{-x} dx + \int_M^\infty x^2 e^{-x} dx$$

$$\le (\text{a number}) + \int_M^\infty \frac{1}{x^2} dx$$

$$< \infty$$

S_N is bounded & increasing $\Rightarrow$ convergent //

Problem 5. Evaluate the integral $\int \ln x dx$.

Use integration by part:

let $u = \ln x$, $dv = dx$

$du = \frac{1}{x} dx$ $v = x$

$$\therefore \int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - x + C$$

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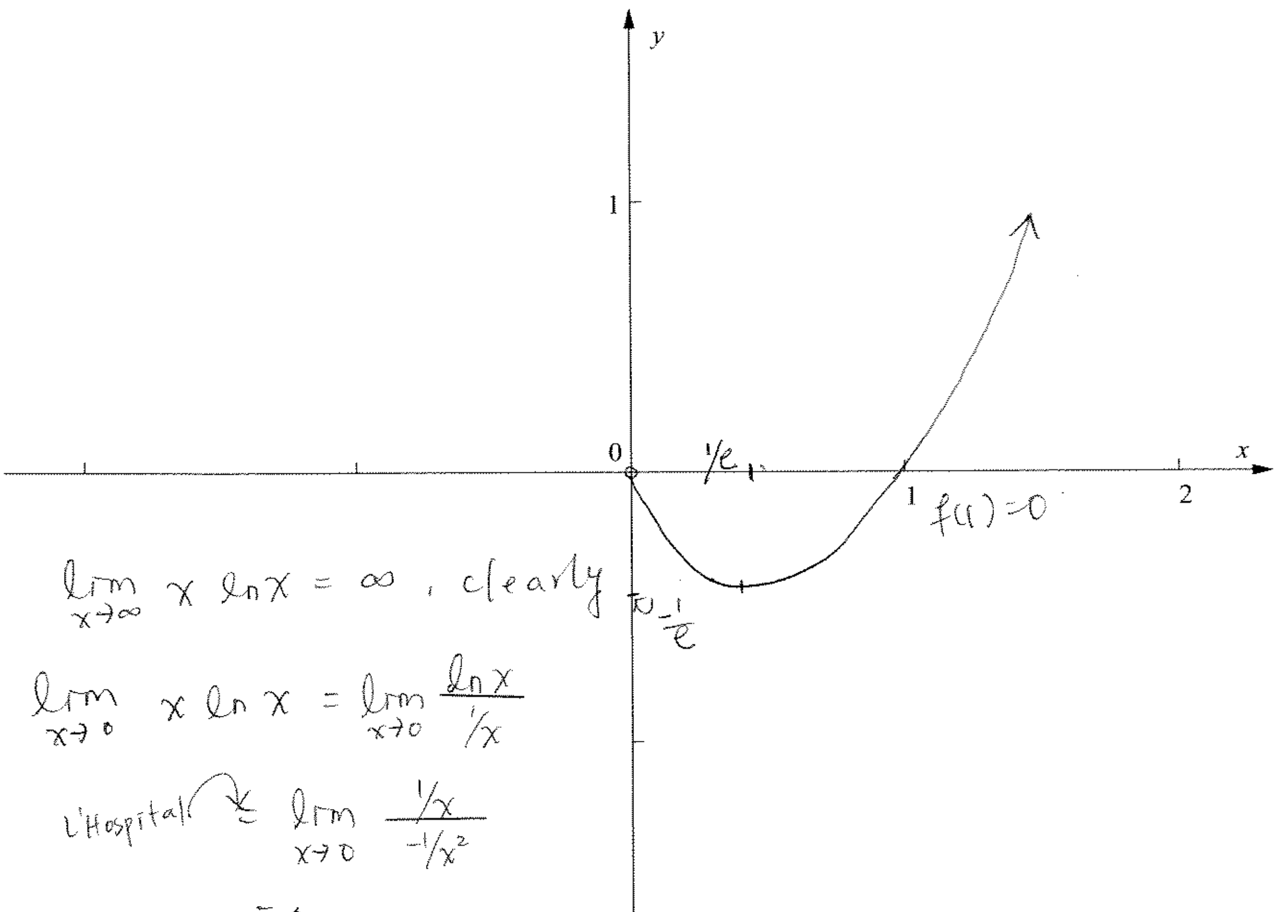
Problem 6. Graph the function $x \ln x$ on the interval $(0, +\infty)$. Indicate minima, maxima, convexity and inflection points, as well as the limits of the function as $x \to 0$ and $x \to +\infty$. Use the coordinate axes below:



$\lim_{x\to\infty} x \ln x = \infty$, clearly

$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x}$

L'Hospital $= \lim_{x\to 0} \frac{1/x}{-1/x^2}$

$= 0$

$f(x) = x \ln x$, $f'(x) = \ln x + 1$, $f''(x) = \frac{1}{x}$

critical points: $f'(x) = 0 \Rightarrow \ln x = -1 \Rightarrow x = \frac{1}{e}$

concavity: $f''(x) = \frac{1}{x} > 0$ for all x in $(0, \infty)$

(use the next page if you need extra space)

f'' + + + + + + + + + + + + ...

f' − − − − − , + + + + + + ...

0 1/e

Decreasing concave up

increasing concave up

(turn page for more problems)

Problem 7. (a) Assume $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$. State a theorem about $\lim_{n\to\infty} A_{n+1}$ and another theorem about $\lim_{n\to\infty}(A_n + B_n)$ and $\lim_{n\to\infty}(A_n - B_n)$.

(b) Use the theorems from (a) to prove that if the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence of terms a_n has limit 0.

ⓐ $\lim_{n\to\infty} A_n = A \Rightarrow \lim_{n\to\infty} A_{n+1} = A$

& $\lim_{n\to\infty} B_n = B \Rightarrow \begin{cases} \lim_{n\to\infty} (A_n + B_n) = A + B. \\ \lim_{n\to\infty} (A_n - B_n) = A - B. \end{cases}$

ⓑ let $\{S_N\}_{N=1}^{\infty}$ be sequence of partial sums

$$S_N = \sum_{n=1}^{\infty} a_n.$$

$\sum_{n=1}^{\infty} a_n$ converges exactly when $S_N \to S$

Moreover, $a_n = S_n - S_{n-1}$

Taking limit on both sides and using the statements from ⓐ

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$$
$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= S - S = 0.$$

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Problem 8. Use a trigonometric substitution to evaluate the integral $\int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr$.

let $r = \sin\theta$, $dr = \cos\theta \, d\theta$.

& $1 - r^2 = 1 - \sin^2\theta = \cos^2\theta$.

& $\begin{cases} r=0 \Rightarrow \theta = 0 \\ r=1 \Rightarrow \theta = \pi/2 \end{cases}$

$$\therefore \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr = \int_0^{\pi/2} \frac{\sin^2\theta}{\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2} d\theta$$

$$= \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right] \Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

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Problem 9. Let $f(x) = \dfrac{x+1}{(x-1)(x-2)(x-3)}$. Use partial fractions to evaluate the integral $\int f(x)dx$.

$$\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$= (A+B+C)x^2 + (-5A-4B-3C)x + 6A+3B+2C$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ -5A-4B-3C=1 \\ 6A+3B+2C=1 \end{cases} \Rightarrow (A,B,C) = (1,-3,2)$$

$$\int f(x)\,dx = \int\left[\frac{1}{x-1} - \frac{3}{x-2} + \frac{2}{x-3}\right]dx$$

$$= \ln|x-1| - 3\ln|x-2| + 2\ln|x-3| + C$$

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Problem 10. Determine the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n+3}$. Be sure to check the convergence at the endpoints of the interval!

$$\rho = \lim_{n\to\infty} \frac{2^n+3}{2^{n+1}+3} |x-1|$$

$$= |x-1| \lim_{n\to\infty} \frac{1+3/2^n}{2+3/2^n}$$

$$= \frac{|x-1|}{2}$$

$\therefore$ Series converges for $|x-1| < 2$,

OR $-1 < x < 3$

at end points,

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n+3} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n+3}$ diverges.

Since $\frac{(-1)^n 2^n}{2^n+3} \not\to 0$

the same thing happen to $x=3$,

where $\sum_{n=0}^{\infty} \frac{2^n}{2^n+3}$ diverges, as $\frac{2^n}{2^n+3} \to 1 \neq 0$

$\therefore$ $|x-1| < 2$, OR $-1 < x < 3$.

radius of convergence: 2.

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Problem 11. Let $S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{\sqrt{k}}$.
By a theorem, the limit $S = \lim_{n\to\infty} S_n$ exists. Find a number n such that

$$|s_n - S| < 1/(25).$$

By estimate of sum of alternating series, $|S_n - S| \le |a_{n+1}|$

$\therefore$ Need to find n s.t. $1/\sqrt{n} \le \frac{1}{25}$

or $\frac{1}{n} < 625$. or $n > 625$

$\therefore$ Any # > 625 will do. //

Problem 12. How many terms do you need to keep in the Taylor series expansion for e^x around zero to approximate the number e to 10 decimal places? You may use that $e < 3$.

Recall Taylor's inequality, (with $a=0$)

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1},$$

M = upper bound of $|f^{(n+1)}(x)|$

at $x=1$, $|f^{(n+1)}(1)| = e < 3$

$\therefore$ it suffices to find n s.t.

$$\frac{3}{(n+1)!} \le 10^{-11}$$

OR $(n+1)! \ge 3 \times 10^{11}$

then we need $n+1 \ge 15$

OR $n \ge 14$

$\therefore$ 14 terms.