MATH 31B LECTURE 1 AND 3

FINAL VERSION B

Solutions.

Please note: Show your work. Except on multiple choice problems, correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted.

#1	#2	#3	#4	#5	#6	
#7	**#8**	**#9**	**#10**	**#11**	**#12**	**Total**

SID:____________ TA: ____________ Section(circle): Tuesday Thursday

Name:____________________

Problem 1. (Multiple choice, 10 pts) Let $f(x) = \sum_{n=0}^{\infty} \frac{2n}{n^2+1} x^n$. Find the value of $f''(0)$. Indicate your answer in the box below.

$\boxed{B}$ (a) 4/5; (b) 8/5; (c) 1/5; (d) 0; (e) None of the above.

$f(x) = 0 \cdot x^0 + \frac{2}{3} x + \frac{4}{5} x^2 +$ terms incl. x^3 or high

$f''(x) = \frac{8}{5} +$ terms incl. x or higher

$f''(0) = \frac{8}{5}$.

Problem 2. (Multiple choice, 20 pts) Let $f(x) = 2x + \cos x$. Find the value of the derivative $(f^{-1})'(\pi)$. Indicate your answer in the box below:

$\boxed{C}$ (a) 1/3; (b) 1/2; (c) 1; (d) 0; (e) None of the above.

$f^{-1}(\pi) = \frac{\pi}{2}$ since $f(\frac{\pi}{2}) = \pi$.

$(f^{-1})'(\pi) = \frac{1}{f'(\pi/2)} = \frac{1}{2 + (-\sin\frac{\pi}{2})} = \frac{1}{1} = 1.$

Problem 3. (Multiple choice, 20 pts) Of the two series

$$(1)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

and

$$(2)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which are convergent? Indicate your answer in the box below:

$\boxed{D}$ (a) both converge; (b) both diverge; (c) (1) converges but (2) diverges;; (d) (2) converges but (1) diverges; (e) None of the above.

(1) diverges (p-series, $p = \frac{1}{2}$)

(2) converges (AST)

Problem 4. (Multiple choice, 20 pts) Which of the following is the Taylor series for $e^{-x}\sin(x^2)$?

$\boxed{E}$

(a) $x + x^2 - x^3/6 - x^4/6 + \cdots$; (b) $x + x^2 - x^3/4 - x^4/12 + \cdots$; (c) $x + x^2 + x^3/2 - x^4/6 + \cdots$; (d) $x + x^2 - x^3/2 - x^4/6 + \cdots$; (e) None of the above.

$$e^{-x}\sin(x^2) = \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots\right)\left(\frac{x^2}{1!} - \frac{x^6}{3!} + \cdots\right)$$

= terms involving x^2 or higher.

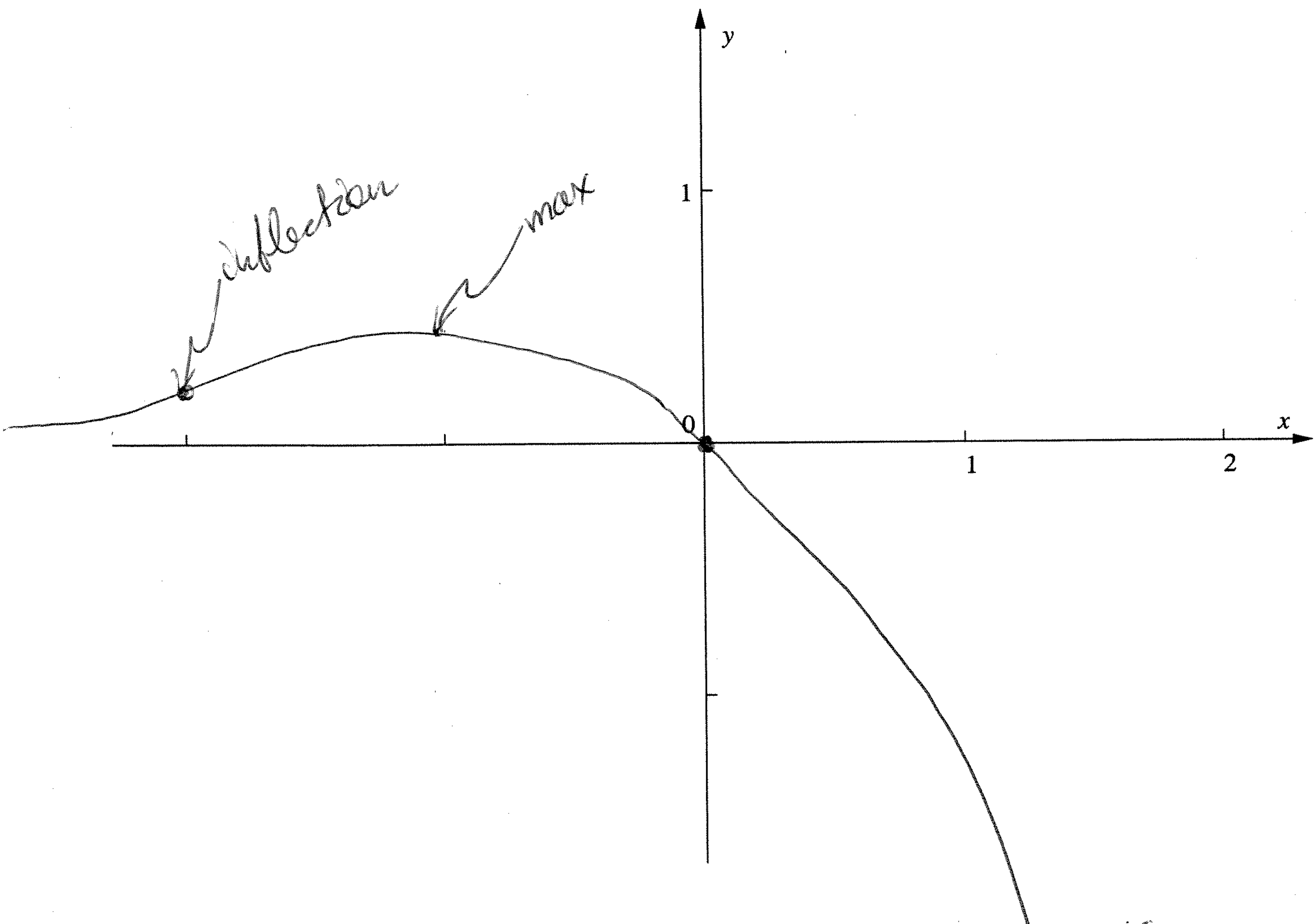
Problem 5. Is the improper integral $\int_0^\infty \frac{\cos^2 x}{1+x^2}$ convergent or divergent? Explain.

Convergent: $\frac{\cos^2}{1+x^2} \leq \frac{1}{1+x^2}$

$$\int_0^\infty \frac{1}{1+x^2} = \lim_{L\to\infty} \int_0^L \frac{1}{1+x^2} =$$

$$= \lim_{L\to\infty} \tan^{-1} L = \pi/2.$$

Problem 6. Graph the function $-xe^x$ on the real axis. Indicate minima, maxima, convexity and inflection points, as well as the limits of the function as $x \to 0$ and $x \to +\infty$. Use the coordinate axes below:



$f(x) = -xe^x$; $f'(x) = -e^x - xe^x = -(1+x)e^x$

$f''(x) = -(2+x)\,e^x$

$f'(x) > 0$ if $x < -1$, $f'(x) < 0$ if $x > -1$

increasing on $(-\infty, -1)$, decreasing on $(-1, \infty)$, max @ -1

(use the next page if you need extra space)

$f''(x) > 0$ if $x < -2$, $f''(x) < 0$ if $x > -2$.

Concave up on $(-\infty, -2)$, Concave down on $(-2, \infty)$, inflection @ -2.

(turn page for more problems)

Asymptotes:

$$\lim_{x\to\infty} -xe^{x} = -\infty \quad \text{since } x\to\infty,\ e^{x}\to\infty$$

$$\lim_{x\to-\infty} -xe^{x} = -\lim_{x\to-\infty} \frac{x}{e^{-x}}$$

$$\overset{\text{L'Hosp.}}{=} -\lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0.$$

So asymptotes: 0 as $x\to-\infty$

$-\infty$ as $x\to+\infty$.

Problem 7. Find a polynomial $p(x)$ with the property that the limit

$$\lim_{x\to 0} \frac{\cos x^3 - p(x)}{x^6}$$

exists. Hint: expand the numerator as a Taylor series.

$\cos x^3 = 1 - \frac{(x^3)^2}{2!}$ + terms of degree x^8 or higher.

we want $\cos x^3 - p(x)$ to be divisible by x^6.

So can take $p(x) = 1$, since then

$\cos x^3 - 1 = -\frac{x^6}{2!}$ + terms of degree x^8 or higher.

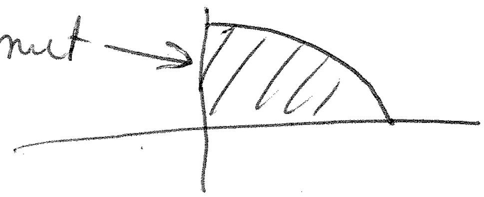
You can actually take $p(x) = 1 + q(x)$,

$q(x)$ polynomial of the form $a_6x^6 + a_7x^7 + \dots + a_nx^7$

for any $n, a_6, \dots, a_n$.

Problem 8. Set up and evaluate an integral that gives the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Area = 4 × area of the segment →



$$\frac{y^2}{4} = 1 - \frac{x^2}{9}, \qquad y^2 = 4 - \frac{4x^2}{9},$$

$$y = \sqrt{4 - \frac{4x^2}{9}} = \frac{2}{3}\sqrt{9 - x^2}$$

$$\text{Area} = 4\int_0^3 \frac{2}{3}\sqrt{9 - x^2}\,dx = \frac{8}{3}\int_0^3 \sqrt{9 - x^2}\,dx.$$

$$x = 3\sin\theta, \quad 0 \le \theta \le \pi/2; \quad d\theta = 3\cos\theta\,d\theta.$$

$$\text{Area} = \frac{8}{3}\int_0^{\pi/2} \sqrt{9 - 9\sin^2\theta}\;3\cos\theta\,d\theta$$

$$= 24\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{24}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= 6\pi.$$

Problem 9. Let $f(x) = \dfrac{x-3}{x(x+1)(x-1)}$. Use partial fractions to evaluate the integral $\int f(x)dx$.

$$\frac{x-3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{A(x^2-1)+B(x^2-x)+C(x^2+x)}{x(x+1)(x-1)}$$

So $\begin{cases} A+B+C=0 \\ -B+C=1 \\ -A=-3 \end{cases} \Rightarrow \begin{cases} A=3 \\ B+C=-3 \\ -B+C=1 \end{cases} \Rightarrow \begin{cases} A=3 \\ B+C=-3 \\ 2C=-2 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=-2 \\ C=-1 \end{cases}$

$$\int f(x)\,dx = \int\left(\frac{3}{x} - \frac{2}{x+1} - \frac{1}{x-1}\right)dx = 3\ln|x| - 2\ln|x+1| - \ln|x-1| + C$$

Problem 10. Determine the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-4)^n}{3^n+1}$. Be sure to check the convergence at the endpoints of the interval!

$$\rho = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{3^{n+1}+1} \cdot \frac{3^n+1}{(x-4)^n} \right| = |(x-4)| \lim_{n\to\infty} \frac{3^n+1}{3^{n+1}+1}$$

$$= |x-4| \lim_{n\to\infty} \frac{3^n \ln 3}{3^{n+1} \ln 3 \cdot 1} = |x-4| \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = |x-4| \lim_{n\to\infty} \frac{1}{3}$$

$$= \left|\frac{x-4}{3}\right|$$

So we want $\frac{|x-4|}{3} < 1$

$\Rightarrow |x-4| < 3 \quad \Rightarrow R=3$

and

$-3 < x-4 < 3$

$1 < x < 7$

at $x=1$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{3^n+1}$

at $x=7$: $\sum_{n=0}^{\infty} \frac{3^n}{3^n+1}$

$\lim_{n\to\infty} \frac{3^n}{3^n+1} = 1 \neq 0$, so both endpoints diverge.

Interval is $(1,7)$.

Problem 11. Let

$$S_N = \sum_{n=1}^{N} (-1)^n \frac{1}{n^3}.$$

By a theorem, the limit $S = \lim_{N\to\infty} S_N$ exists. Find a number N such that

$$|S_N - S| < \frac{1}{1000}.$$

Justify your answer.

$|S_N - S| = |N^{th} \text{ error}| \leq \frac{1}{(1+N)^3}$ by alternating series error estimation thm.

So we want $\frac{1}{(1+N)^3} < \frac{1}{1000}$

$\Rightarrow (1+N)^3 > 1000$

$\Rightarrow (1+N) > (1000)^{1/3} = 10$

So let $N = 10$

Problem 12. Find the integral $\int_0^\infty e^{-x}\sin x\,dx$. Hint: Integrate by parts on $[0, A]$ with A very large.

$$\int_0^\infty e^{-x}\sin x\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\sin x\,dx.$$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x\,dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$$

$u = e^{-x}$ $dv = \sin x\,dx$
$du = -e^{-x}dx$ $v = -\cos x$

$u = e^{-x}$ $dv = \cos x\,dx$
$du = -e^{-x}dx$ $v = \sin x$

$$\Rightarrow 2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + c$$

$$\Rightarrow \int e^{-x}\sin x\,dx = \frac{-e^{-x}(\sin x + \cos x)}{2} + c$$

$$\lim_{A\to\infty}\int_0^A e^{-x}\sin x\,dx = \lim_{A\to\infty}\left(-\frac{e^{-x}(\sin x + \cos x)}{2}\right)\Big|_0^A$$

$$= \lim_{A\to\infty}\left(\frac{-e^{-A}(\sin A + \cos A)}{2} + \frac{\cos 0}{2}\right) = \lim_{A\to\infty}\frac{-e^{-A}}{2}(\sin A + \cos A) + \frac{1}{2}$$

$$\boxed{= \frac{1}{2}}$$

$$-2 \le \sin A + \cos A \le 2$$

$$e^{-A} \ge \frac{-e^{-A}(\sin A + \cos A)}{2} \ge -e^{-A}$$

$e^{-A} \searrow 0$, $-e^{-A} \searrow 0$

$\lim_{A\to\infty} e^{-A} = 0$

$\lim_{A\to\infty} -e^{-A} = 0$

, so by Squeeze Theorem,

$$\lim_{A\to\infty}\frac{-e^{-A}}{2}(\sin A + \cos A) = 0, \text{ so}$$