**MATH 31B LECTURE 1 AND 3**

**FINAL** VERSION A

A: A A C E

B: B C D E

**Please note:** Show your work. Except on multiple choice problems, correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted.

| #1 | #2 | #3 | #4 | #5 | #6 | |
|---|---|---|---|---|---|---|
| | | | | | | |
| #7 | #8 | #9 | #10 | #11 | #12 | Total |
| | | | | | | |

SID:__________ TA: __________ Section(circle): Tuesday Thursday

Name: Liu, Chih-Chung

**Problem 1.** (Multiple choice, 10 pts) Let $f(x) = \sum_{n=0}^{\infty} \frac{2n}{n^2+1} x^n$. Find the value of $f''(0)$. Indicate your answer in the box below.

$\boxed{A}$ (a) 8/5; (b) 1/5; (c) 4/5; (d) 0; (e) None of the above.

$$f''(0) = 2!\, a_2$$
$$= 2a_2$$
$$= 2 \cdot \frac{4}{4+1} = \frac{8}{5}$$

**Problem 2.** (Multiple choice, 20 pts) Let $f(x) = 2x - \cos x$. Find the value of the derivative $(f^{-1})'(\pi)$. Indicate your answer in the box below:

$\boxed{A}$ (a) 1/3; (b) 1/2; (c) 1; (d) 0; (e) None of the above.

$$(f^{-1})'(\pi) = \frac{1}{f'(f^{-1}(\pi))}$$

$2x + \cos x = \pi$.

$x = \frac{\pi}{2}$

$$= \frac{1}{f'(\frac{\pi}{2})}$$

$$= \frac{1}{(2 + \sin x)\Big|_{\frac{\pi}{2}}} = \frac{1}{3}$$

**Problem 3.** (Multiple choice, 20 pts) Of the two series

$$(1)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

and

$$(2)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which are convergent? Indicate your answer in the box below:

$\boxed{c}$ (a) both converge; (b) both diverge; (c) (1) converges but (2) diverges;; (d) (2) converges but (1) diverges; (e) None of the above.

(1) converges (alternating series test.)

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \quad \& \quad \frac{1}{\sqrt{n}} \to 0.$$

(2) diverges. p-series with $p = \frac{1}{2} < 1$

**Problem 4.** (Multiple choice, 20 pts) Which of the following is the Taylor series for $e^x \sin(x^2)$?

$\boxed{E}$

(a) $x + x^2 - x^3/6 - x^4/6 + \cdots$; (b) $x + x^2 - x^3/4 - x^4/12 + \cdots$; (c) $x + x^2 + x^3/2 - x^4/6 + \cdots$; (d) $x + x^2 - x^3/2 - x^4/6 + \cdots$; (e) None of the above.

$$e^x \sin(x^2)$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{} + \frac{x^5}{120} - \cdots\right)$$
$$= x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} + \frac{x^4}{6} \cdots$$
$$= x + x^2 + \frac{x^3}{3} \cdots$$

$$e^x \sin x^2$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots\right)$$
$$= x^2 + x^3 + \frac{x^4}{2} + \cdots$$

**Problem 5.** Is the improper integral $\int_0^\infty \frac{\sin^2 x}{1+x^2} dx$ convergent or divergent? Explain.

Let $S_N = \int_0^N \frac{\sin^2 x}{1+x^2}\, dx$

1 $\{S_N\}$ is monotonically increasing, since

$$S_{N+1} - S_N = \int_N^{N+1} \frac{\sin^2 x}{1+x^2}\, dx \geq 0 \quad \left( \frac{\sin^2 x}{1+x^2} \geq 0 \right)$$

It's bounded above:

$$S_N = \int_0^N \frac{\sin^2 x}{1+x^2}\, dx \leq \int_0^\infty \frac{\sin^2 x}{1+x^2}\, dx$$

$$\leq \int_0^\infty \frac{1}{1+x^2}\, dx$$

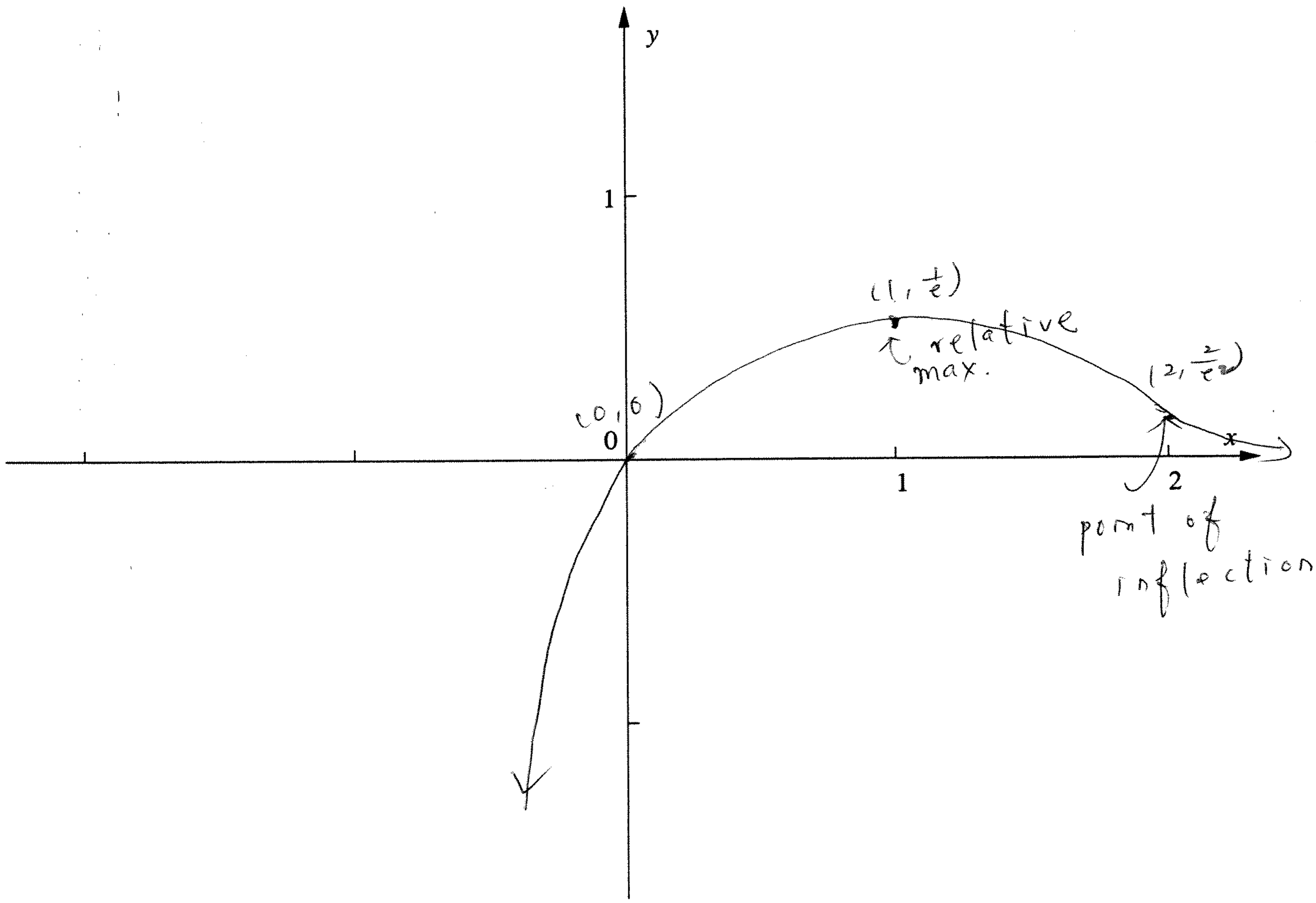
$$= \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

$\therefore$ $\{S_N\}$ is convergent

$\Rightarrow$ $\int_0^\infty \frac{\sin^2 x}{1+x^2}\, dx$ is convergent.

2 $\cancel{\int_0^\infty \frac{\sin^2 x}{1+x^2}\, dx \leq \int}$

**Problem 6.** Graph the function $xe^{-x}$ on the real axis. Indicate minima, maxima, convexity and inflection points, as well as the limits of the function as $x \to \cancel{0}$ $-\infty$ and $x \to +\infty$. Use the coordinate axes below:




$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{\text{Hospital}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} xe^{-x} = -\infty$$

$$\cancel{= \lim_{x\to-\infty} \frac{x}{e^x} = \lim_{x\to-\infty} \frac{1}{e^x} \to \infty}$$

$f(x) = xe^{-x}$

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$

$f'(x) > 0$ when $x < 1$, $f'(x) < 0$ when $x > 1$

$f'(x) = 0 \iff x = 1$

$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(x-2)$, $= 0 \iff x = 2$

$f''(x) > 0$ when $x > 2$, $f''(x) < 0$ when $x < 2$

(use the next page if you need extra space)

| | | | | |
|---|---|---|---|---|
| $f''$ | concave down → − − − | | con. up + + + | |
| $f'$ | increase + + + + | max | decrease − − − | |
| $x$ | ↗ | 1 | ↘ | 2 |

(turn page for more problems)

**Problem 7.** Find a polynomial $p(x)$ with the property that the limit

$$\lim_{x\to 0} \frac{\cos x^3 - p(x)}{x^6}$$

exists. Hint: expand the numerator as a Taylor series.

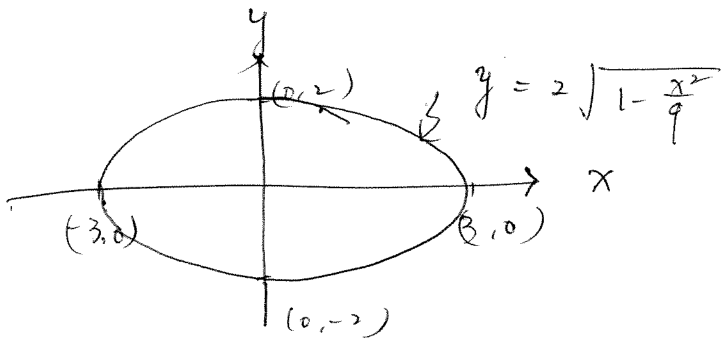
$$\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$= 1 - \frac{x^6}{2} + \frac{x^{12}}{4!} - \cdots$$

The limit exists only when the numerator is a power series with terms that have degree $\geqslant 6$.

$\therefore$ Simply let $p(x) = 1$.

**Problem 8.** Set up and evaluate an integral that gives the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.




$$\text{Area} = 2\int_{-3}^{3} 2\sqrt{1-\frac{x^2}{9}}\,dx$$

$$= 4\int_{-3}^{3} \sqrt{1-\frac{x^2}{9}}\,dx \quad ; \quad \text{let } x = 3\sin\theta$$

$$1-\frac{x^2}{9} = 1-\sin^2\theta = \cos^2\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot 3\cos\theta\,d\theta$$

$$x=3 \Rightarrow \theta = \frac{\pi}{2}$$

$$x=-3 \Rightarrow \theta = -\frac{\pi}{2}$$

$$= 12\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$dx = 3\cos\theta\,d\theta$$

$$= 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= 6\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 6\pi.$$

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**Problem 9.** Let $f(x) = \dfrac{x+3}{x(x-1)(x+1)}$. Use partial fractions to evaluate the integral $\int f(x)dx$.

$$\frac{x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\Rightarrow A(x-1)(x+1) + B\,x(x+1) + C\,x(x-1) = x+3$$

$$//$$

$$(A+B+C)x^2 + (B-C)x + (-A)$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ B-C=1 \\ -A=3 \end{cases} \Rightarrow (A,B,C) = (-3,2,1)$$

$$\therefore f(x) = \frac{-3}{x} + \frac{2}{x-1} + \frac{1}{x+1}$$

$$\int f(x)\,dx = -3\ln|x| + 2\ln|x-1| + \ln|x+1| + C \quad //$$

**Problem 10.** Determine the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$. Be sure to check the convergence at the endpoints of the interval!

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{2^{n+1}+1}}{\frac{1}{2^n+1}} |x-3|$$

$$= |x-3| \lim_{n\to\infty} \frac{2^n+1}{2^{n+1}+1}$$

$$= |x-3| \lim_{n\to\infty} \frac{1+1/2^n}{2+1/2^n} = \frac{|x-3|}{2}$$

$$\rho < 1 \Rightarrow \frac{|x-3|}{2} < 1 \Rightarrow -2 < x-3 < 2$$

OR. $1 < x < 5$

Endpoint:

$x = 5$, $\sum_{n=0}^{\infty} \frac{2^n}{2^n+1}$ diverges,

Since $\frac{2^n}{2^n+1} \to 1 \neq 0$.

$x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n+1}$ also diverges.

Since $\frac{(-1)^n 2^n}{2^n+1} \not\to 0$

$\therefore$ interval of convergence : $|x-3| < 2$, radius : 2

**Problem 11.** Let

$$S_N = \sum_{n=1}^{N} (-1)^n \frac{1}{n^2}.$$

By a theorem, the limit $S = \lim_{N\to\infty} S_N$ exists. Find a number $N$ such that

$$|S_N - S| < \frac{1}{100}.$$

Justify your answer.

$$|S_N - S| < b_{N+1} = \frac{1}{(N+1)^2}$$

$\therefore$ it suffices to choose an $N$

s.t $\frac{1}{(N+1)^2} < \frac{1}{100}$, or $N+1 > 10$.

or. $N > 9$.

$\therefore$ for example, take $N = 10$. //

**Problem 12.** Find the integral $\int_0^\infty e^{-x}\cos x dx$. Hint: Integrate by parts on $[0, A]$ with $A$ very large.

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\cos x\,dx$$

let $u = e^{-x}$ ; $dv = \cos x\,dx$

$du = -e^{-x}dx$ ; $v = \sin x$

$$\int_0^A e^{-x}\cos x\,dx = e^{-x}\sin x\Big|_0^A + \int_0^A e^{-x}\sin x\,dx$$

Again, let $u = e^{-x}$, $dv = \sin x\,dx$

$du = -e^{-x}dx$, $v = -\cos x$

$$= e^{-x}\sin x\Big|_0^A - e^{-x}\cos x\Big|_0^A - \int_0^A e^{-x}\cos x\,dx.$$

$$\therefore\ 2\int_0^A e^{-x}\cos x\,dx$$

$$= e^{-A}\sin A - e^{-A}\cos A + 1$$

OR $\int_0^A e^{-x}\cos x\,dx$

$$= \frac{e^{-A}\sin A - e^{-A}\cos A + 1}{2}$$

$$\therefore\ \lim_{A\to\infty}\int_0^A e^{-x}\cos x\,dx = \frac{1}{2}$$

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