# MATH 31B SECTION 2
# SECOND MIDTERM

**Please note:** Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of the professor within 2 weeks of the exam.*

| #1 | #2 | #3 | #4 | #5 | Total |
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SID:____________________ TA: ________________ Section(circle): Tuesday Thursday

Name:______________________________________________

**Problem 1.** Evaluate the definite integral

$$\int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx.$$

*Solution.* Using $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we get

$$\sin^2 x \, \cos^2 x = \frac{1}{4}(1-\cos 2x)(1+\cos 2x) = \frac{1}{4}(1-\cos^2 2x) = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1-\cos 4x).$$

Hence the integral is the same as

$$\int_0^{\pi/2} \frac{1}{8}(1-\cos 4x) dx = \frac{\pi}{16}.$$

**Problem 2.** Find the limit

$$\lim_{t\to 0^+} t\ln(\sin t).$$

*Solution.* Applying L'Hopital's rule, we have

$$\begin{aligned}\lim_{t\to 0^+} t\ln(\sin t) &= \lim_{t\to 0^+} \frac{\ln \sin t}{1/t} \\ &= \lim_{t\to 0^+} \frac{\frac{\cos t}{\sin t}}{-1/t^2} \\ &= \lim_{t\to 0^+} \frac{t^2}{\tan t}.\end{aligned}$$

Applying L'Hopital's rule again gives

$$\lim_{t\to 0^+} \frac{t^2}{\tan t} = \lim_{t\to 0^+} \frac{2t}{\sec^2 t} = \frac{0}{1} = 0.$$

**Problem 3.** Use a trigonometric substitution to evaluate the definite integral:

$$\int_0^{\sqrt{3}} \frac{x^3}{(x^2+1)^{3/2}} dx.$$

*Solution.* Let $x = \tan\theta$, so that $dx = \sec^2\theta d\theta$. Since $0 \le x \le \sqrt{3}$, $0 \le \theta \le \pi/3$. Thus the integral is the same as

$$\begin{aligned}
\int_0^{\pi/3} \frac{\tan^3\theta}{(\sqrt{\tan^2\theta+1})^3} \sec^2\theta d\theta &= \int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta} d\theta \\
&= \int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta} d\theta \\
&= \int_0^{\pi/3} \frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta} d\theta.
\end{aligned}$$

Let $u = \cos\theta$, $du = -\sin\theta d\theta$, so that the integral is equal to

$$\begin{aligned}
-\int_1^{1/2} \frac{1-u^2}{u^2} du &= \int_{1/2}^{1} (\frac{1}{u^2} - 1) du \\
&= (-\frac{1}{u} - u)\Big|_{1/2}^{1} \\
&= -2 + 1/2 + 2 = 1/2.
\end{aligned}$$

**Problem 4.** Evaluate the indefinite integral $\int \sin(\ln x)\; dx$.

*Solution*. Let $I = \int \sin\ln x\; dx$. We integrate by parts with $u = \sin\ln x$ and $dv = dx$. Thus $du = \frac{1}{x}\cos\ln x$ and $v = x$. Hence

$$I = x\sin\ln x - \int \cos(\ln x)\; dx.$$

Integrating by parts again with $u = \cos\ln x$, $v = dx$, $du = -\frac{1}{x}\sin\ln x$ and $v = x$, we get

$$I = x\sin\ln x + x\cos\ln x - \int \sin\ln x\; dx = x\sin\ln x - x\cos\ln x + C - I.$$

Solving for $I$ gives that

$$I = \frac{1}{2}(x\sin\ln x - x\cos\ln x) + C.$$