MATH 31B SECTION 2
SECOND PRACTICE MIDTERM

Problem 1. Evaluate the definite integral

$$\int_0^{\pi/2} \sin^2 x \cos 2x \, dx.$$

Solution. Using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ we get

$$\sin^2 x \ \cos 2x = \frac{1}{2}(1 - \cos 2x)\cos 2x = \frac{1}{2}\cos 2x - \frac{1}{2}\cos^2 2x.$$

Using the identity $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$ gives

$$\sin^2 x \ \cos 2x = \frac{1}{2}\cos 2x - \frac{1}{4}(1 + \cos 4x).$$

Hence

$$\begin{aligned} \int_0^{\pi/2} \sin^2 x \ \cos 2x \, dx &= \int_0^{\pi/2} \left(\frac{1}{2}\cos 2x - \frac{1}{4} - \frac{1}{4}\cos 4x\right) dx \\ &= -\frac{\pi}{8}. \end{aligned}$$

Problem 2. Find the limit

$$\lim_{x\to 0^+} x^x.$$

Solution. We have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x},$$

since the function e^x is continuous.

Now,

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0^+} \frac{1/x}{-(1/x^2)} = \lim_{x\to 0^+} -x = 0.$$

Thus

$$\lim_{x\to 0^+} x^x = e^0 = 1.$$

Problem 3. Use a trigonometric substitution to evaluate the definite integral:

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx.$$

*Solution.*We let $x = \sin\theta$, so that $0 \leq \theta \leq \pi/2$ and $dx = \cos\theta d\theta$. The integral then becomes

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin^2\theta\ \cos\theta d\theta}{\sqrt{1-\sin^2\theta}} &= \int_0^{\pi/2} \sin^2\theta d\theta = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)d\theta \\ &= \frac{\pi}{4} - \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} = \pi/4.\end{aligned}$$

Problem 4. Evaluate the definite integral

$$\int_0^1 \frac{r^2}{\sqrt{1+r^2}}dr.$$

Solution. We let $r = \tan\theta$, so that $0 \le \theta \le \pi/4$ and $d\theta = \sec^2\theta d\theta$. Hence the integral is the same as

$$\begin{aligned}
\int_0^{\pi/4} \frac{\tan^2\theta}{\sec\theta}\sec^2\theta d\theta &= \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^3\theta}d\theta = \int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos^3\theta}d\theta \\
&= \int_0^{\pi/4}(\sec^3\theta - \sec\theta)d\theta \\
&= \int_0^{\pi/4}\sec^3\theta d\theta - \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} \\
&= \int_0^{\pi/4}\sec^3\theta d\theta - \ln(\sqrt{2}+1).
\end{aligned}$$

To evaluate

$$\int_0^{\pi/4}\sec^3\theta d\theta,$$

we use integration by parts. Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. Then $du = \sec\theta\tan\theta d\theta$ and $v = \tan\theta$. Hence

$$\begin{aligned}
\int_0^{\pi/4}\sec^3\theta d\theta &= \sec\theta\ \tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4}\tan^2\theta\sec\theta d\theta \\
&= \sqrt{2} - \int_0^{\pi/4}(\sec^2\theta - 1)\sec\theta d\theta \\
&= \sqrt{2} - \int_0^{\pi/4}\sec^3\theta d\theta - + \int \sec\theta d\theta \\
&= \sqrt{2} + \ln(\sqrt{2}+1) - \int_0^{\pi/4}\sec^3\theta d\theta.
\end{aligned}$$

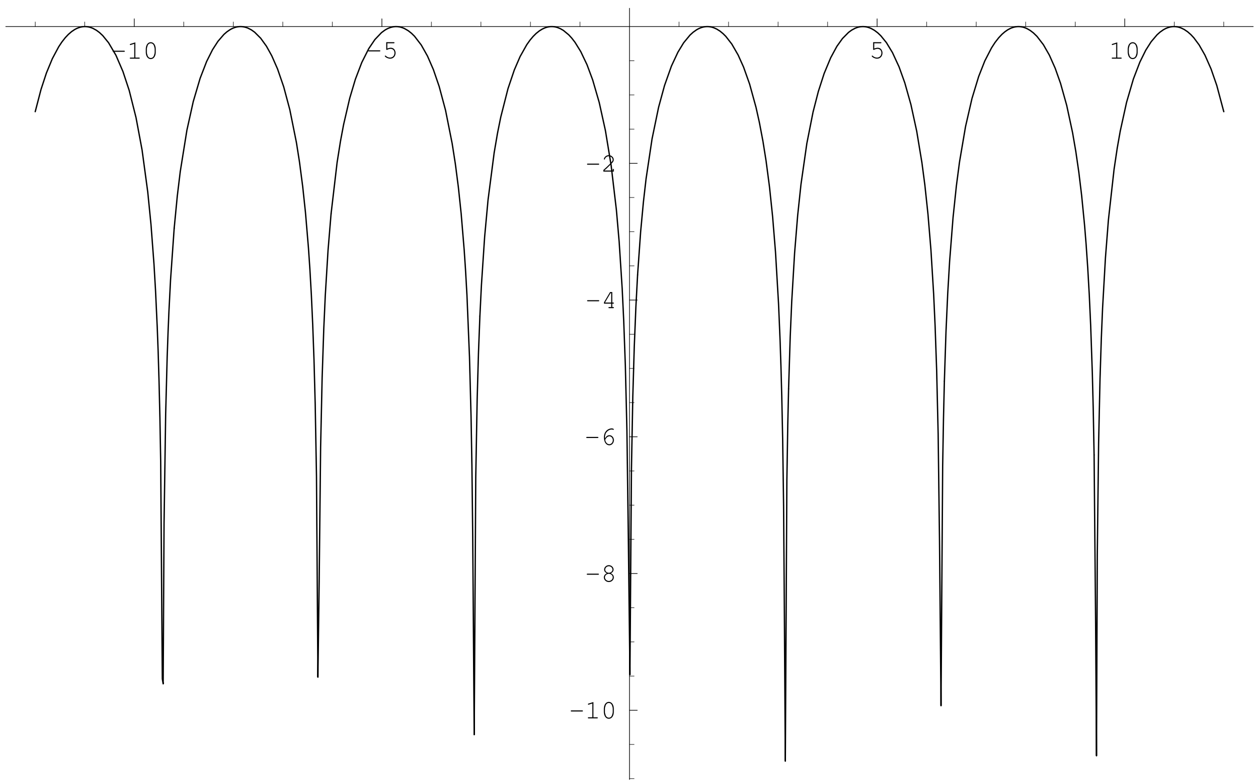
Solving for the integral $\int_0^{\pi/4}\sec^3\theta d\theta$ gives

$$\int_0^{\pi/4}\sec^3\theta d\theta = \frac{1}{2}(\sqrt{2}+\ln(\sqrt{2}+1)).$$

Hence the final answer is

$$\frac{1}{2}(\sqrt{2}-\ln(\sqrt{2}+1))$$

Problem 5. Sketch the graph of the function $f(x) = \ln\left(\sin^2 x\right)$. Indicate the limits at infinity, vertical asymptotes, maxima, minima and inflection points. Please use the coordinate axes drawn below.



Solution. Let $f(x) = \ln(\sin^2 x)$. Then $f'(x) = \frac{2\sin x \cos x}{\sin^2 x} = 2\cot x$ and $f''(x) = -2\csc^2 x$. It follows that $f''(x) < 0$ for all x and hence the function is concave down and there are no inflection points. Also, $f'(x) = 0$ at $x = \frac{\pi}{2} + n\pi$, $n = 0, \pm1, \pm2, \ldots$, where the function takes on the value of $\ln 1 = 0$. Since $\sin^2 x \leq 1$, it follows that $f(x) \leq 0$ for all x and hence these points are the maxima. Lastly, since $|f'(x)| \to \infty$ when $x \to n\pi$, $n = 0, \pm1, \pm2, \ldots$ and $f(x) \to -\infty$ when $x \to n\pi$, $n = 0, \pm1, \pm2, \ldots$, $f(x)$ has vertical asymptotes at these points.

Problem 6. Use integration by parts to evaluate the definite integral

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr.$$

Solution. We let $u = r^2$, $dv = \dfrac{rdr}{\sqrt{4+r^2}}$. Hence $v = \sqrt{4+r^2}$ and $du = 2rdr$. The integral is then

$$r^2\sqrt{4+r^2}\Big|_0^1 - \int_0^1 2r\sqrt{4+r^2}dr = \sqrt{5} - \int_0^1 2r\sqrt{4+r^2}dr.$$

Substituting $u = 4 + r^2$ gives

$$\begin{aligned}
\sqrt{5} - \int_4^5 \sqrt{u}du &= \sqrt{5} - \frac{2}{3}u^{3/2}\Big|_4^5 \\
&= \sqrt{5} - \frac{2}{3}5\sqrt{5} + \frac{2}{3}(\sqrt{4})^3 \\
&= \frac{-7\sqrt{5}}{3} + \frac{16}{3}.
\end{aligned}$$