MATH 31B SECTION 2
FIRST MIDTERM

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of the professor within 2 weeks of the exam.*

#1	#2	#3	#4	#5	Total

SID:____________________ TA: ________________ Section(circle): Tuesday Thursday

Name:____

Problem 1. Let S be the sphere of radius 1. Set up and then evaluate the integral expressing the volume of S using (a) the method of cross-sections and (b) the method of cylindrical shells.

Solution. (a) Consider the plane parallel to the y,z coordinate plane and intersecting the x-axis at the point x. The intersection of this plane with the interior of the sphere is a disk

$$y^2+z^2=1-x^2$$

of radius $\sqrt{1-x^2}$. Hence its area $A(x)$ is $\pi(1-x^2)$ and the volume of the sphere is

$$\int_{-1}^{1} A(x)dx = \int_{-1}^{1} \pi(1-x^2)dx = \pi(x-\frac{1}{3}x^3)\Big|_{-1}^{1} = \frac{4}{3}\pi.$$

(b) The shpere can be obtained by rotating the half-circle $y=\pm\sqrt{1-x^2}$ about the y-axis. If we fix a value of x and rotate the resulting line, we get a cylinder of radius x and height $2\sqrt{1-x^2}$. If we let x vary between 0 and 1, the resulting cylinders combine to fill up the whole sphere. Thus the volume is

$$\int_0^1 2\pi x\cdot 2\sqrt{1-x^2}dx.$$

Substituting $u=1-x^2$, $du=-2xdx$ gives us

$$V=\int_1^0 -2\pi\sqrt{u}du = 2\pi\int_0^1 \sqrt{u}du = \frac{4}{3}\pi u^{3/2}\Big|_0^1 = \frac{4}{3}\pi.$$

Problem 2. Evaluate the indefinite integral $\int e^{x+e^x}\,dx$.

Solution. See practice midterm.

Problem 3. Consider the definite integral

$$I = \int_1^3 \sqrt{x} dx.$$

For which values of n can you guarantee that the Midpoint Rule would approximate the integral to within $\frac{1}{3} \cdot 10^{-4}$? Justify your answer.

Solution. Denote the error by E. We know that

$$|E| \le \frac{K(b-a)^3}{24n^2} = \frac{K(3-1)^3}{24n^2} = \frac{K}{3n^2},$$

where K is the maximum value of the absolute value of the second derivative of $f(x) = \sqrt{x}$ on $[1,3]$. We compute $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$. We see that $f''(x)$ is monotone increasing on $[1,3]$ and therefore its maximum is attained at $x = 3$ and its minimum is attained at $x = 1$.Since $|f''(3)| = \frac{1}{4\sqrt{27}}$ and $|f''(1)| = \frac{1}{4}$, we see that

$$|f''(x)| \le \frac{1}{4}, \qquad x \in [1,3].$$

Hence we can take $K = \frac{1}{4}$ and our estimate becomes

$$|E| \le \frac{1}{4 \cdot 3n^2} = \frac{1}{12n^2}.$$

We thus want to see for which n the inequality

$$\frac{1}{12n^2} \le \frac{1}{3}10^{-4}$$

is satisfied. Rewriting, we have

$$\begin{aligned} 1 &\le 12 \cdot \frac{1}{3} \cdot 10^{-4} n^2 = 4 \cdot 10^{-4} n^2, \\ n^2 &\ge \frac{10^4}{4}, \\ n &\ge \sqrt{\frac{10^4}{4}} = \frac{100}{2} = 50. \end{aligned}$$

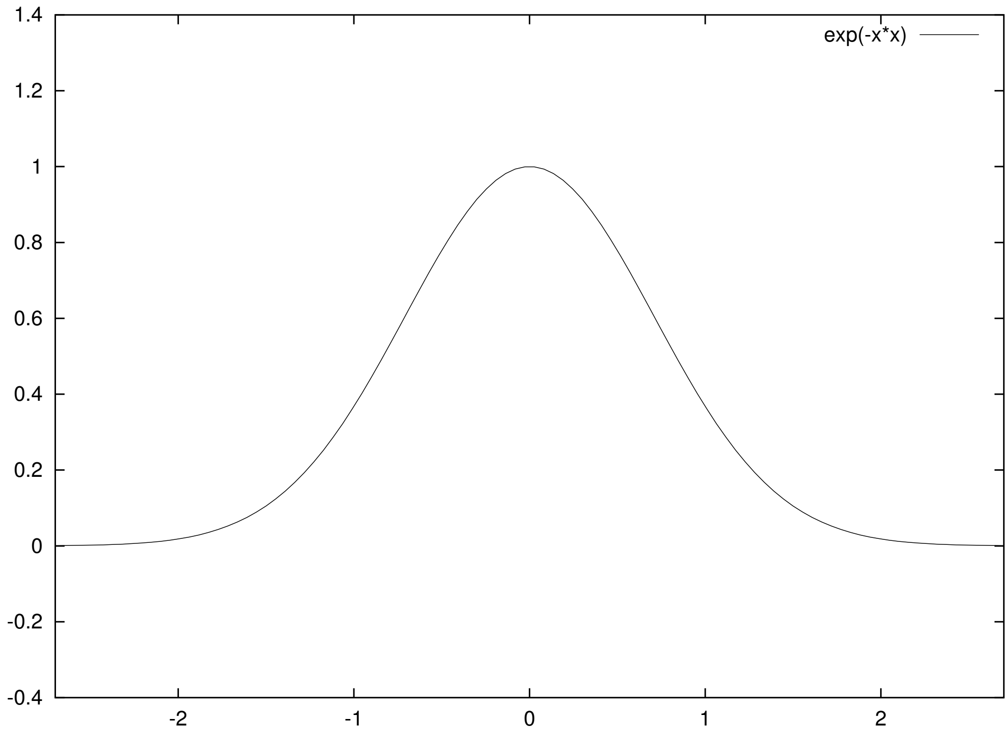
Thus $n \ge 25$.

Problem 4. Let $f(x) = x + \frac{1}{2}\cos x$. (a) Show that the function f is monotone and conclude that it is 1-1. (b) Find the value of the derivative $(f^{-1})'(\pi - \frac{1}{2})$.

Solution. (a) Note that $f'(x) = 1 - \frac{1}{2}\sin x \geq \frac{1}{2} > 0$. Thus f is strictly increasing and hence is 1-1. (b) We note that $f(\pi) = \pi - \frac{1}{2}$, so that $f^{-1}(\pi - \frac{1}{2}) = \pi$. Hence

$$(f^{-1})'(\pi - \frac{1}{2}) = \frac{1}{f'(f^{-1}(\pi - \frac{1}{2}))} = \frac{1}{f'(\pi)} = \frac{1}{1 - \frac{1}{2}\sin\pi} = 1.$$

Problem 5. Sketch the graph of the function $f(x) = e^{-x^2}$. Indicate the limits at infinity, maxima, minima and inflection points. Please use the coordinate axes drawn below.



Solution. The graph is the "bell curve".

We have that $f'(x) = -2xe^{-x^2}$, which is zero at $x = 0$. So there is only one extreme point. Also, the function is increasing on $(-\infty, 0)$ and is decreasing $(0, +\infty)$, since the first derivative is postitive on $(-\infty, 0)$ and negative on $(0, +\infty)$.

Since $f''(x) = -2e^{-x^2} - 2x(-2xe^{-x^2}) = (-2+4x^2)e^{-x^2}$, we see that $f''(x) = 0$ if $4x^2 = 2$, i.e., $x = \pm\sqrt{2}$. Thus the inflection points are at $x = \sqrt{2}$ and $x = -\sqrt{2}$.

Since $f''(0) < 0$, it follows that the only extreme point is a maximum.

On the interval $(-\infty, -\sqrt{2})$, $f''(x) > 0$, so the function is concave up. On the interval $(-\sqrt{2}, \sqrt{2})$, $f''(x) < 0$ and so the function is concave down. Finally, on the interval $(\sqrt{2}, +\infty)$ the second derivative is again positive, and the function is concave up.

Lastly, as $x \to \pm\infty$, $-x^2 \to -\infty$ and so $e^{-x^2} \to 0$. Thus

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$