MATH 31B SECTION 2
FIRST PRACTICE MIDTERM

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted. *If you have a question about the grading or believe that a problem has been graded incorrectly, you must bring it to the attention of the professor within 2 weeks of the exam.*

Problem 1. Let S be the cone obtained by rotating the line triangle bounded by the lines $y = x$, $x = 1$, $y = 0$ about the x-axis. Set up and then evaluate the integral expressing the volume of S using (a) the method of cross-sections and (b) the method of cylindrical shells.

Solution. (a) Let us fix the value of x. The cross-section of the cone by the plane parallel to the y,z plane and intersecting the x-axis at x is a disk of radius x (see Figure 1a). Therefore, the area of the cross-section

$$A(x) = \pi x^2.$$

We then find that the volume is

$$V = \int_0^1 A(x)dx = \int_0^1 \pi x^2 dx = \frac{1}{3}\pi.$$

(b) The cone can be seen to be filled with cylinders of radii r varying from 0 to 1; the height of the cylinder of radius r is $1 - r$ (see Figure 1b). We then get

$$V = \int_0^1 2\pi r(1-r)dr = \int_0^1 2\pi r - 2\pi r^2 dr = (1 - \frac{2}{3})\pi = \frac{1}{3}\pi.$$

Problem 2. Let S be the region consisting of those points (x,y,z) that lie inside of the sphere of radius 1, and which satisfy $0 \leq x \leq 0.3$. Find the volume of S.

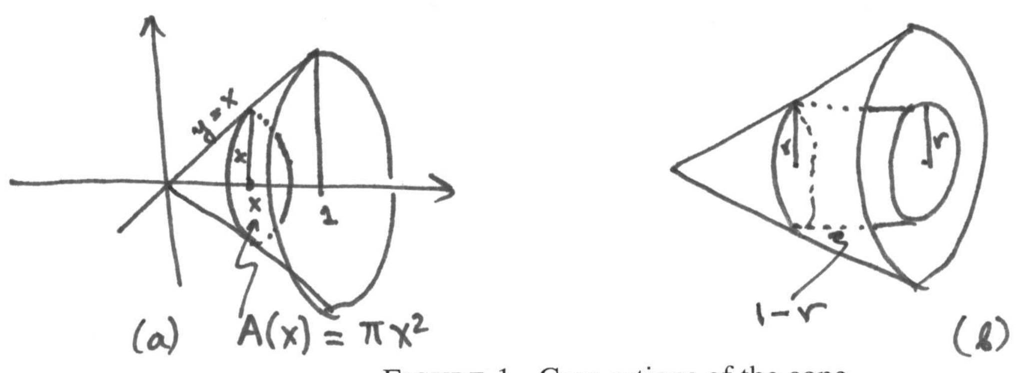

FIGURE 1. Cros-setions of the cone.

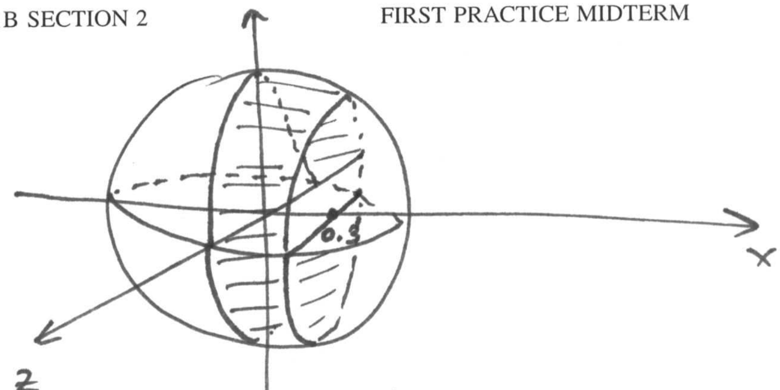

FIGURE 2. The sphere.

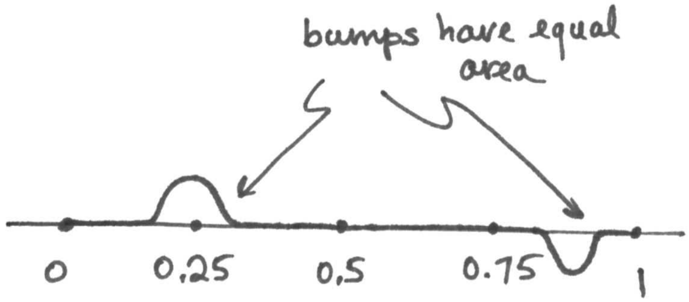

FIGURE 3. The function.

Solution. We'll use the method of cross-sections. Consider the cross-section of S by a plane parallel to the y,z coordinate place and intersecting the x axis at x. The cross-section is a circle of radius $\sqrt{1-x^2}$ (see Figure 2). The area $A(x)$ of this cross-section is $\pi(\sqrt{1-x^2})^2 = \pi(1-x^2)$. Since $0 \leq x \leq 0.3$, the volume is given by

$$V = \int_0^{0.3} \pi(1-x^2)dx = 0.3\pi - \frac{1}{3}(0.3)^2\pi = \pi(0.3+0.009) = 3.009\pi.$$

Problem 3. Give an example of a function $f(x)$ for which the trapezoidal rule approximation with $n=2$ to the integral $\int_0^1 f(x)dx$ is exact (i.e., the error is zero), while the trapezoidal rule approximation to the same integral with $n=4$ gives a nonzero error. (No formula is necessary; a graph of the function, with explanations, will suffice).

Solution. The trapezoid rule with $n=2$ involves dividing the interval into 2 subintervals and approximating the function by straight lines on these two subintervals. Thus this approximation only cares about $f(0)$, $f(0.5)$ and $f(1)$ (i.e., the values of f at the endpoints of these two intervals).

The trapezoid rule with $n=4$ involves doing the same, but over 4 intervals. So the approximation now also looks at $f(0.25)$ and $f(0.75)$.

Now the idea is to make a function twhich is zero, except for two "bumps". The bumps are chosen so that (1) two areas of the bumps cancel each other, so that the total intergral of f is zero; (2) f is zero at the ends and the midpoint of the interval; (3) f is also zero at 0.75, but f is strictly positive at 0.25. See Figure 3.

Because of (1), the total intergral of f is zero.

Because of (2), the values of f at the endpoints of the intervals involved in the midpoint rule for $n = 2$ are zero, and so the approximation gives zero for the value of the interval.

Because of (3), the trapezoids involved in the approximation of the graph of f have area zero on the intervals $[0.5, 0.75]$ and $[0.75, 1]$, but have a strictly positive area on both $[0, 0.25]$ (where the top of the trapezoid connects $0 = f(0)$ and $f(0.25) > 0$) and $[0.25, 0.5]$ (where the top connects $f(0.25) > 0$ and $f(0.5) = 0]$. So the trapezoid rule would give a strictly positive answer instead of zero.

Problem 4. Compute the integral $\int e^{x+e^x} dx$.

Solution. Using the formula $e^{a+b} = e^a e^b$ we get

$$\int e^{x+e^x} dx = \int e^x e^{e^x} dx.$$

Substituting $u = e^x$, $du = e^x dx$ gives us

$$\int e^x e^{e^x} dx = \int e^u du = e^u + C = e^{e^x} + C.$$

Problem 5. Let $f(x) = \sqrt{x^2+1}$, $x \geq 0$. (a) Show that f is one-to-one on $[0, +\infty)$; (b) compute the inverse of $f(x)$; (c) find the derivative of the inverse of $f(x)$ at the point $\sqrt{2}$.

Solution. (a) The function x^2 is strictly increasing on $[0, +\infty)$; hence so are the functions $x^2 + 1$ and $\sqrt{x^2+1}$. It follows that f is 1-1. (b) We have

$$\begin{aligned} y &= \sqrt{x^2+1} \\ y^2 &= x^2+1 \\ x^2 &= y^2-1 \\ x &= \pm\sqrt{y^2-1}. \end{aligned}$$

Now, since $x \geq 0$, we know that we must take the plus sign, and so

$$x = \sqrt{y^2-1}.$$

Thus

$$f^{-1}(x) = \sqrt{x^2-1}.$$

(c) Note that $f(1) = \sqrt{2}$, so $f^{-1}(\sqrt{2}) = 1$. Since $f'(x) = \frac{x}{\sqrt{x^2+1}}$, we have that

$$f'(\sqrt{2}) = \frac{1}{f'(f^{-1}(\sqrt{2}))} = \frac{\sqrt{1^2+1}}{1} = \sqrt{2}.$$

(You can also compute the derivative of the inverse directly in this case).

Problem 6. Graph the function $f(x) = e^x + e^{-x}$.

Solution. The graph is presented below (Figure 4). We have that $f'(x) = e^x - e^{-x}$, and so $f(x) \geq 0$ if $x \geq 0$ and $f(x) \leq 0$ if $x \leq 0$. Hence f is decreasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$. Since $f''(x) = e^x + e^{-x} > 0$ (since $e^x > 0$ for all x) we find that there are no inflection points and that f is concave up. Finally, $\lim_{x\to+\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$, we find that $\lim_{x\to\pm\infty} f(x) = +\infty$.

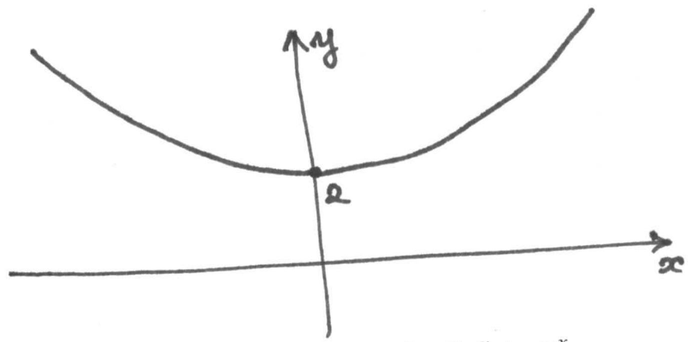

FIGURE 4. The graph of $e^x + e^{-x}$.