MATH 31B SECTION 2
FINAL EXAM SOLUTIONS.

Please note: Show your work. Correct answers not accompanied by sufficent explanations will receive little or no credit. Please call one of the proctors if you have any questions about a problem. No calculators, computers, PDAs, cell phones, or other devices will be permitted.

#1	#2	#3	#4	#5	#6	
#7	#8	#9	#10	#11	#12	Total

SID:____________________ TA: ________________ Section(circle): Tuesday Thursday

Name:____

Problem 1. Let $f(x) = \sum_{n=0}^{\infty} \frac{3n}{2n+5} x^n$. Find the value of $f'''(0)$.

Solution. If c_n denotes the coefficient of x^n in this power series, then $f'''(0) = 3! \cdot c_3 = 3! \cdot \frac{3 \cdot 3}{2 \cdot 3 + 5} = \frac{54}{11}$.

Problem 2. Determine if the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ is convergent or divergent. If it is convergent, find its sum.

Solution. We have that

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

The series is telescoping:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n+1} - \frac{1}{n+2} = 1 - \frac{1}{n+2} \to 1.$$

Hence the sum of the series is 1.

Problem 3. Let $f(x) = \tan^{-1} x$. Find a power series representation for f around 0. (*Hint:* represent $\tan^{-1} x$ as an integral).

Solution. We have:

$$\begin{aligned}
\tan^{-1} x &= \int \frac{1}{1+x^2} dx + C \\
&= \int \frac{1}{1-(-x^2)} dx + C \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} + C \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C.
\end{aligned}$$

To find C, we evaluate both sides at $x = 0$; since $\tan^{-1} 0 = 0$, $C = 0$ and

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Problem 4. Is the improper integral $\displaystyle\int_0^\infty e^{\sin x}dx$ convergent or divergent? Explain. *Hint:* $\sin x$ is periodic.

Solution. Since the function $\sin x$ is periodic, we have that

$$\int_0^{2\pi} e^{\sin x}dx = \int_{2\pi}^{4\pi} e^{\sin x}dx = \int_{2n\pi}^{2(n+1)\pi} e^{\sin x}dx,$$

for all n. Let L be a number, and let n be the largest integer so that $2n\pi < L$. Since $e^{\sin x} \geq 0$ for all x, we have that $\int_{2n\pi}^L e^{\sin x}dx \geq 0$, and so

$$\begin{aligned}\int_0^L e^{\sin x}dx &= \int_0^{2n\pi} e^{\sin x}dx + \int_{2n\pi}^L e^{\sin x}dx \geq \int_0^{2n\pi} e^{\sin x}dx \\ &= \int_0^{2\pi} e^{\sin x}dx + \int_{2\pi}^{4\pi} e^{\sin x}dx + \cdots + \int_{2(n-1)\pi}^{2n\pi} e^{\sin x}dx \\ &= n\int_0^{2\pi} e^{\sin x}dx \geq (L-2\pi)\int_0^{2\pi} e^{\sin x}dx,\end{aligned}$$

by periodicity and since $L - 2\pi \leq n$ (because n is the largest integer with $2n\pi < L$). Since $e^{\sin x} > 0$, it follows that the integral $\int_0^{2\pi} e^{\sin x}dx$ has some strictly positive value, say A. We conclude that

$$\int_0^L e^{\sin x}dx \geq (L-2\pi)A.$$

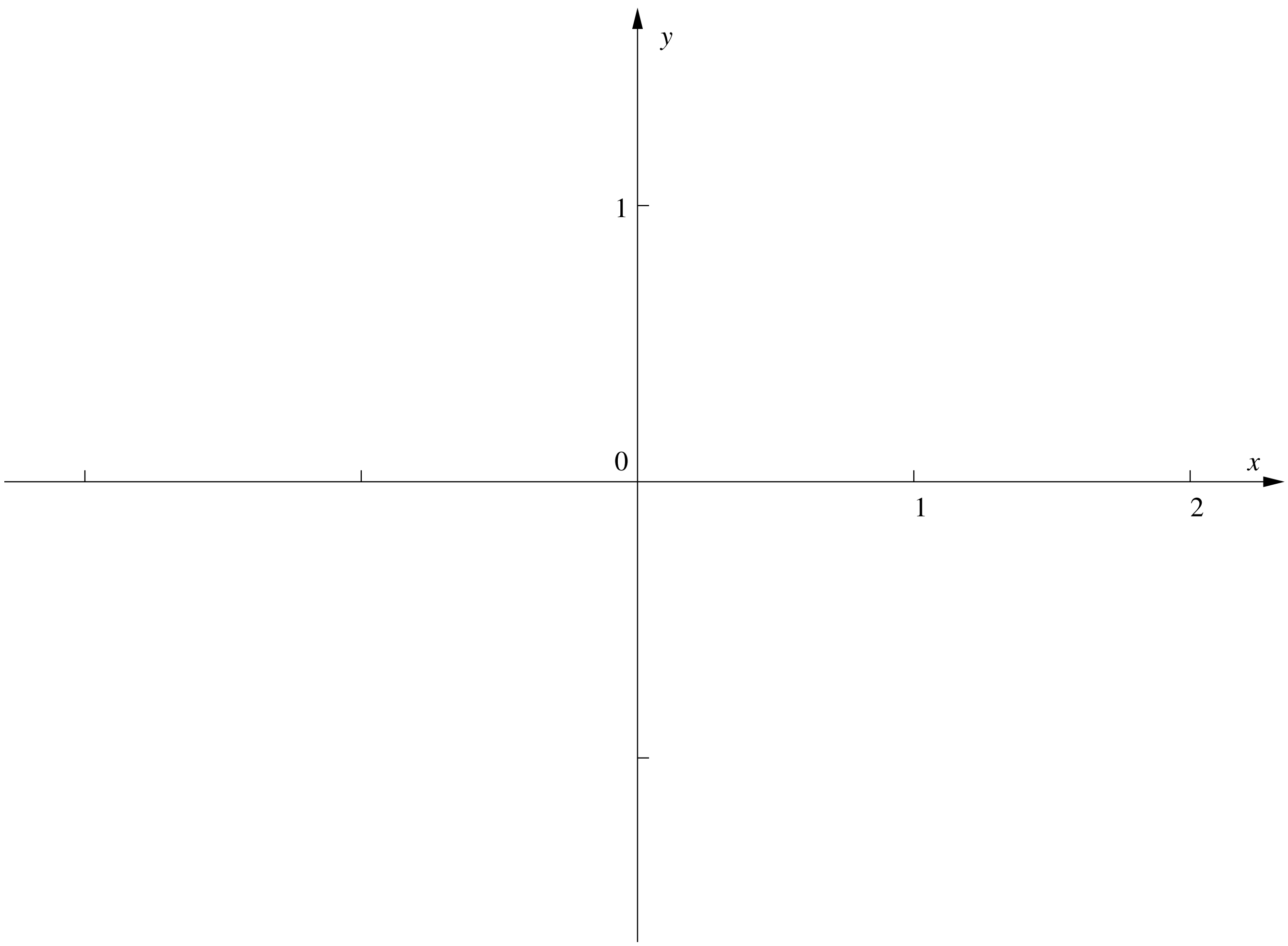
Since $\lim_{L\to\infty}(L-2\pi)A = \infty$, it follows that also $\lim_{L\to\infty}\int_0^L e^{\sin x}dx = \infty$. Hence the integral diverges.

Problem 5. . Evaluate the integral $\int \ln x dx$.

Solution. We integrate by parts with $u = \ln x$, $dv = dx$, $du = \frac{1}{x}dx$, $v = x$:

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C.$$

Problem 6. Graph the function $x \ln x$ on the interval $(0, +\infty)$. Indicate minima, maxima, convexity and inflection points, as well as the limits of the function as $x \to 0$ and $x \to +\infty$. Use the coordinate axes below:



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Solution. We have:

$$f'(x) = \ln x + 1$$
$$f''(x) = \frac{1}{x}.$$

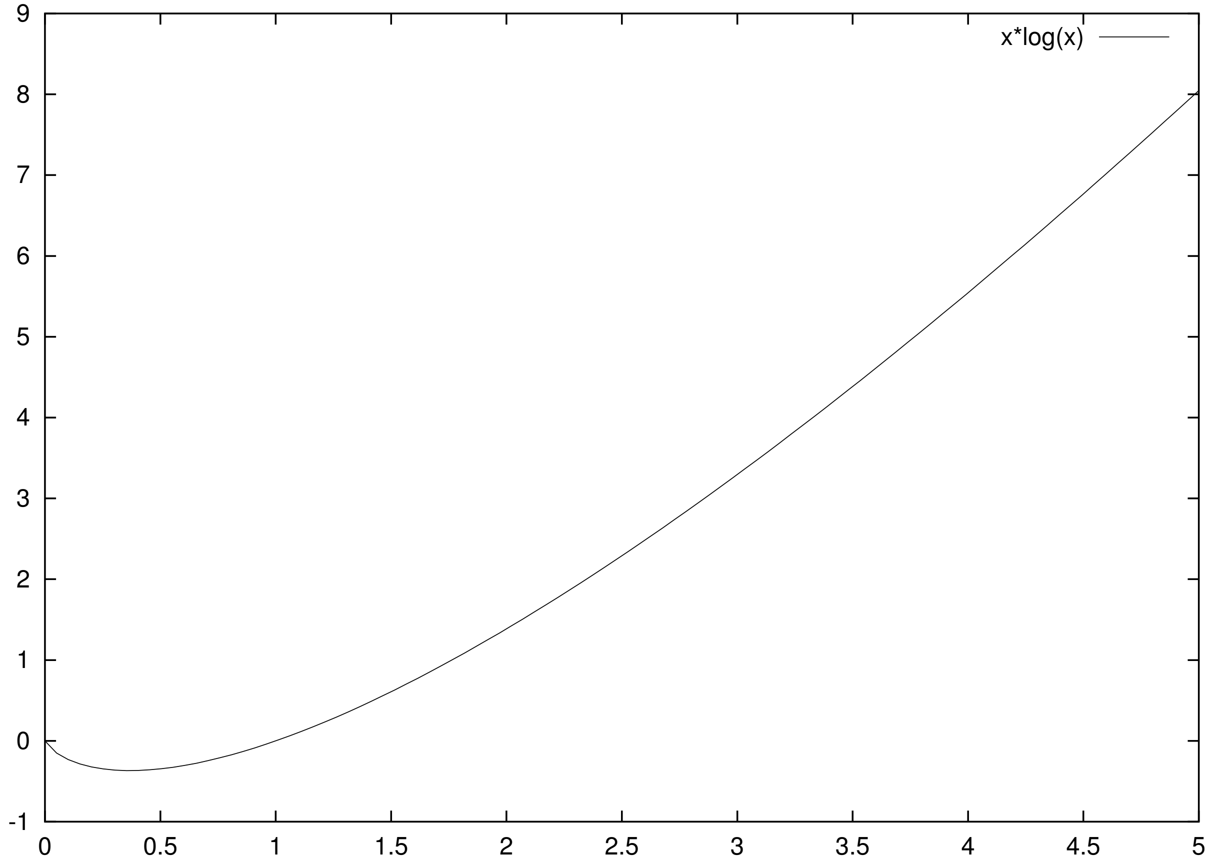
Since $f''(x) > 0$ for all x, it follows that the function is concave up. We have that $f'(x) < 0$ if $\ln x < -1$, i.e., $x < 1/e$ and $f'(x) > 0$ if $x > 1/e$. Thus the function is decreasing on $[0, 1/e]$ and is increasing on $[1/e, +\infty)$. We need to find the limits of the function as $x \to 0$ and $x \to +\infty$. We have

$$\lim_{x\to\infty} x \ln x = \infty,$$

since both x and $\ln x \to \infty$ as $x \to \infty$. For the other limit, we use L'Hospital's rule:

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} x = 0.$$

It follows that the function has a minimum of $-1/e$ at $x = 1/e$ and no maximum. Its graph is presented below:


x*log(x)
9
8
7
6
5
4
3
2
1
0
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Problem 7. State the definition of a one-to-one function. Prove that the function $f(x) = x^3 + 4$ is one-to-one.

Solution. A function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. For our function $f(x) = x^3 + 4$, assume that $f(x_1) = f(x_2)$. Thus $x_1^3 + 4 = x_2^3 + 4$, so that $x_1 = x_2$. Hence f is one-to-one.

Problem 8. Use a trigonometric substitution to evaluate the integral $\int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr$.

Solution. Let $r = \sin\theta$, $dr = \cos\theta d\theta$. The integral becomes:

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta} \cos\theta d\theta &= \int_0^{\pi/2} \sin^2\theta = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2\theta) d\theta \\ &= \pi/4 - \frac{1}{4}(\sin 2\theta)\Big|_0^{\pi/2} = \pi/4. \end{aligned}$$

Problem 9. Let $f(x) = \dfrac{x+1}{(x-1)(x-2)(x-3)}$. Use partial fractions to evaluate the integral $\int f(x)dx$.

Solution. We try

$$f(x) = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

We get:

$$x + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Letting $x = 1$ gives $A = 1$; letting $x = 2$ gives $B = -3$; letting $x = 3$ gives $C = 2$. Hence

$$f(x) = \frac{1}{x-1} + \frac{-3}{x-2} + \frac{2}{x-3}.$$

It follows that

$$\int f(x)dx = \log(x-1) - 3\log(x-2) = 2\log(x-3) + C.$$

Problem 10. Determine the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n+3}$.

Solution. Using the ratio test gives us:

$$L = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}+3} \cdot \frac{2^n+3}{(x-1)^n} \right| = |x-1| \cdot \frac{1}{2}.$$

Thus the series is convergent for $|x-1| < 2$ and divergent for $|x-1| > 2$. Hence the radius of convergence is 2. We check endpoints: when $x = 3$ or $x = -1$, the series of absolute values is $\sum \frac{1}{2^n+3}$, which is convergent by comparison to $\sum \frac{1}{2^n}$ (convergent geometric series). So the series is absolutely convergent both at $x = 3$ and $x = -1$. Thus the interval of convergence is $[-1, 3]$.

Problem 11. Find the limit of the sequence $a_n = \left(1 + \frac{2}{n}\right)^n$.

Solution. We have

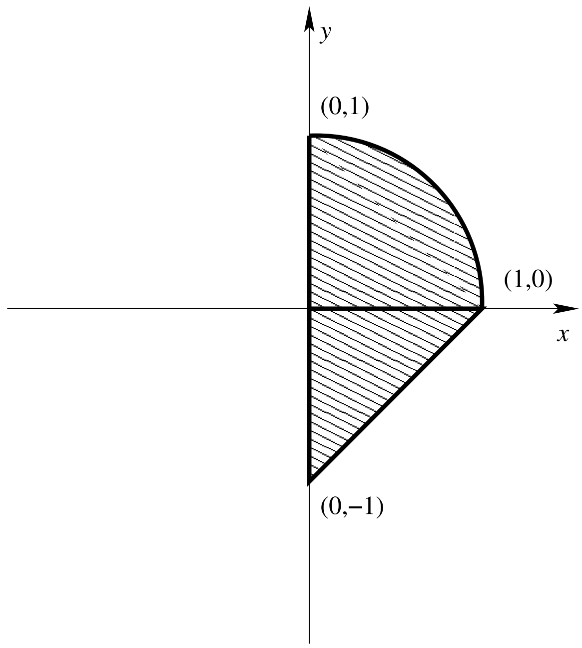
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln(a_n)} = \lim_{n\to\infty} e^{n\ln(1+2/n)}.$$

Let $f(x) = x\ln(1 + 2/x)$. By L'Hospital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{-2/x^2}{1+2/x}}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2.
\end{aligned}$$

Hence $\lim a_n = e^2$.

Problem 12. Find the volume of the "ice cream cone" obtained by rotating about the y axis the region bounded by the y axis, the quarter-circle $y = \sqrt{1-x^2}$, $x, y \geq 0$, and the triangle with vertices $(0,0)$, $(1,0)$ and $(0,-1)$:



Solution. The cylindrical shell of radius x has height $\sqrt{1-x^2} - (x-1)$. Hence the volume is:

$$\begin{aligned}
\int_0^1 2\pi x(\sqrt{1-x^2} - (x-1))dx &= \int_0^1 2\pi x\sqrt{1-x^2}dx - \int_0^1 2\pi x(x-1)dx \\
&= (-\frac{2}{3}\pi(1-x^2)^{3/2} - \frac{2}{3}\pi x^3 + \pi x^2)\Big|_0^1 \\
&= \pi/3 + 2\pi/3 = \pi.
\end{aligned}$$