MATH 31B/2 PRACTICE FINAL EXAM SOLUTIONS.

Please note: the aim of this practice final is to give you several problems on the material not covered by the first two midterms and practice midterms. ***You should therefore treat the first two midterms and practice midterms as a part of the practice final.***

(1) Let $f(x) = \sin x$. Find n so that Taylor's polynomial of degree n around 0 approximates $\sin(1)$ to within 10^{-2}. Justify your answer.

Solution. Using Taylor's remainder formula, if we keep n terms in Taylor's polynomial, the error is at most

$$E_n \leq \frac{M(x-a)^{n+1}}{(n+1)!},$$

where M is the maximal value of the $n+1$-st derivative of $\sin(x)$ on the interval $[0,1]$. Hence $M = 1$ and

$$E_n \leq \frac{1^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}.$$

Hence we want n so that $\frac{1}{(n+1)!} \leq \frac{1}{100}$, i.e., $(n+1)! \geq 100$. Since $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 < 100$ and $5! = 5 \cdot 4! = 120 > 100$, we get that $n = 4$ works.

(2) Let

$$f(x) = \frac{2x+4}{x^3-1}.$$

Express $f(x)$ as a sum of terms using partial fractions. Use this to evaluate $\int f(x)dx$.

Solution. Since $x^3 - 1 = (x-1)(x^2+x+1)$, we will try to find A, B, C so that

$$\frac{2x+4}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

We then have

$$2x+4 = A(x^2+x+1) + (Bx+C)(x-1) = x^2(A+B) + x(A-B+C) + (A-C).$$

Equating coefficients, we get $0 = A+B$, $2 = A-B+C$ and $4 = A-C$. From the first equation we get that $B = -A$ and from the last that $C = A-4$. Substituting this into the second equation gives $2 = A-(-A)+A-4 = 3A-4$. Hence $3A = 6$ and $A = 2$. Thus $B = -2$ and $C = -2$. We conclude that

$$\frac{2x+4}{x^3-1} = \frac{2}{x-1} + \frac{-2x-2}{x^2+x+1}.$$

To evaluate the integral, we need to evaluate the integrals $\int \frac{2}{x-1}dx = 2\ln(x-1) + C$ and $\int \frac{-2x-2}{x^2+x+1}dx$. In the latter integral, we complete the square: $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4} =$

$\frac{3}{4}((\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}})^2+1)$. Hence the integral becomes

$$\int \frac{-2(x+1)}{\frac{3}{4}((\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}})^2+1)}dx.$$

If we let $y=\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}$,we have that $dy=\frac{2}{\sqrt{3}}xdx$ and $x=\frac{\sqrt{3}}{2}y-\frac{1}{2}$; hence the integral becomes

$$\begin{aligned}\int \frac{-2(\sqrt{3}y+1)}{\sqrt{3}(y^2+1)}dy &= -2\int \frac{y}{y^2+1}dy-\int \frac{2}{\sqrt{3}(y^2+1)}dy\\ &= -\ln(y^2+1)-\frac{2}{\sqrt{3}}\tan^{-1}(y)+C\\ &= -\ln(\frac{3}{4}(x^2+x+1))-\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})+C\\ &= -\ln(x^2+x+1)-\frac{2\tan^{-1}(2x+1)/\sqrt{3})}{\sqrt{3}}+C-\ln(3/4).\end{aligned}$$

Hence the final answer is

$$\int \frac{2x+4}{x^3-1}dx=\ln(x^2+x+1)-\frac{2\tan^{-1}(2x+1)/\sqrt{3})}{\sqrt{3}}+2\ln(x-1)+C.$$

(3) Find the Taylor series for the function $f(x)=\ln|x-1|$. Determine its radius and interval of convergence.

Solution. Since $\ln|x-1|=\int \frac{1}{x-1}dx=-\int \frac{1}{1-x}dx=-\int(\sum x^n)dx$, we can integrate term-by-term to obtain

$$\ln|x-1|=-\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}=-\sum_{n=0}^{\infty}\frac{x^n}{n}.$$

The ratio test gives us

$$L=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|=|x|.$$

Thus the series is convergent if $|x|<1$ and divergent if $|x|>1$. Thus the radius of convergence is 1. We check for convergence at the endpoints. If $x=1$, we get $-\sum\frac{1}{n}$, which is divergent; if $x=-1$, we get $-\sum(-1)^n\frac{1}{n}$, which is convergent by the alternating series test. Hence the interval of convergence is $[-1,1)$.

(4) Find the power series representation of the function $\dfrac{1}{(1-x)^2}$. Determine its radius and interval of convergence.

Solution. We note that

$$\frac{1}{(1-x)^2}=-\frac{d}{dx}\frac{1}{1-x}=-\frac{d}{dx}\sum_{n=0}^{\infty}x^n.$$

Differentiating term-by-term gives us

$$\frac{1}{(1-x)^2} = -\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n-1)x^n.$$

Using the ratio test gives us

$$L = \lim_{n\to\infty} \left| \frac{nx^{n+1}}{(n-1)x^n} \right| = |x|,$$

so the series is convergent for $|x| < 1$ and divergent for $|x| > 1$. Hence the radius of convergence is 1. Checking at endpoints, if $x = \pm 1$, the n-th term has absolute value $n-1$ and thus does not go to zero; the series is therefore divergent by the test for divergence. It follows that the interval of convergence is $(-1, 1)$.

(5) Find the limit of the sequence $a_n = n^{1/n}$.

Solution. Using the logarithm trick, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}.$$

Let $f(x) = \frac{1}{x}\ln x$. Then by L'Hospital's rule, we get

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Hence $\lim \frac{1}{n}\ln n = 0$. Thus $\lim a_n = e^0 = 1$.

(6) Is the improper integral $\int_0^{\infty} \sin(e^x)dx$ convergent or divergent? Explain.

Solution. Let $t = e^x$, $dt = e^x dx$. Then the integral becomes

$$\int_1^{\infty} \frac{1}{t}\sin t dt.$$

Integrating by parts with $u = \frac{1}{t}$ and $dv = \sin t dt$, $du = -\frac{1}{t^2}dt$ and $v = \cos t dt$, we get

$$\int_1^{\infty} \frac{1}{t}\sin t dt = \lim_{L\to\infty} (\frac{1}{t}\cos t \Big|_1^L) + \int_1^{\infty} \frac{1}{t^2}\cos t dt = -\cos 1 + \int_1^{\infty} \frac{1}{t^2}\cos t dt.$$

Now we must decide if $\int_1^{\infty} \frac{1}{t^2}\cos t dt$ is convergent or divergent. If the integral $\int_1^{\infty} \left|\frac{1}{t^2}\cos t\right| dt$ is convergent, so is the integral $\int_1^{\infty} \frac{1}{t^2}\cos t dt$. Since $\left|\frac{1}{t^2}\cos t\right| \le \frac{1}{t^2}$, and the integral $\int_1^{\infty} \frac{1}{t^2}dt = \lim_{L\to\infty} -\frac{1}{L} + 1$ exists, we know that $\int_1^{\infty} \left|\frac{1}{t^2}\cos t\right| dt$ is convergent by the comparison test. Hence the original integral is convergent. (If you go on to learn some complex analysis, you will be able to compute that $\int_{-\infty}^{\infty} \sin(e^x)dx = \pi/2$, so actually our integral must be less than that.)

(7) Find the surface area of the surface of revolution obtained by rotating the parabola $y = x^2$, $0 \le x \le 1$, about the y-axis.

Solution. We have

$$S = \int_0^1 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 2\pi x\sqrt{1+4x^2}dx.$$

Letting $u = 1+4x^2$, we have that $du = 8xdx$, so that

$$S = \int_1^5 \frac{\pi}{4}\sqrt{u}du = \frac{\pi}{6}u^{3/2}\Big|_1^5 = \frac{\pi}{6}(5\sqrt{5}-1).$$

(8) Use the arclength formula to find the length of the circle of radius 1.

Solution. The formula $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$ gives us a semicircle. We compute $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$ so that $1+(dy/dx)^2 = 1+x^2/(1-x^2) = 1/(1-x^2)$. Hence

$$L = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x\Big|_{-1}^1 = \pi/2+\pi/2 = \pi.$$

Thus the length of the circle is 2π.

(9) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent: (a) $\sum(-1)^n\frac{1}{n+\sin(n)}$; (b) $\sum(-1)^n\frac{1}{n^2+\sin(n)}$; (c) $\sum e^{-n}$.

Solution. (a) Let $f(x) = \frac{1}{x+\sin x}$. Then $f'(x) = -\frac{1+\cos x}{(x+\sin x)^2} \leq 0$; furthermore, the derivative is zero only at $x = \pi+2k\pi$. It follows that on each interval $[n, n+1]$, $f'(x) < 0$ except possibly at one point; hence $f(n+1) < f(n)$. Since the series is alternating and for $n \geq 1$, $0 \leq \frac{1}{n+\sin n} \leq \frac{1}{n-1} \to 0$, we have that the series is convergent by the alternating series test. The series is conditionally convergent: if we consider the series of absolute values, $\sum|\frac{(-1)^n}{n+\sin n}| = \sum\frac{1}{n+\sin n}$, we get that $\frac{1}{n+\sin n} \geq \frac{1}{n+1}$; since $\sum\frac{1}{n+1}$ is divergent, the series of absolute values is also divergent, by the comparison test.

(b) Since $\frac{1}{n^2+\sin(n)}/\frac{1}{n^2} \to 1$ as $n \to \infty$and the series $\sum\frac{1}{n^2}$ is convergent, it follows that the series of absolute values $\sum\frac{1}{n^2+\sin n}$ is also convergent, by the limit comparison test. Hence the series in (b) is absolutely convergent.

(c) The series is a geometric series with $r = e^{-1} = 1/e < 1$ and is therefore absolutely convergent.