MATH 132
PRACTICE MIDTERM

Problem 1. State the book's definition of:
(a) The functions $\sin z$, $\cos z$, Log(z) (principal branch), $\sqrt{z}$ (Principal branch), e^z
(b) Continuity of a function at z_0
(c) Analytic function

Solution: see book.

Problem 2. Show that if a function $f(z)$ is analytic, it is continuous. Prove that the function $f(z) = \cos z$ is analytic (use Cauchy-Riemann equations). Conclude that $f(z)$ is continuous.

Solution: See book, theorem on p. 43.
We have that

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

We now verify the Cauchy Riemann equations with $u = \cos x \cos y$ and $v = -\sin x \sinh y$:

$$\begin{aligned} \partial u/\partial x &= -\sin x \cos y = \partial v/\partial y \\ \partial u/\partial y &= \cos x \sinh y = -\partial v/\partial x. \end{aligned}$$

Problem 3. For each of the following functions, find the image of the indicated regions: $f(z) = \frac{1+z}{1-z}$, $g(z) = \sqrt[4]{z}$(principal branch), $h(z) =$ Arg(z): (a) The unit disk $|z| \leq 1$; (b) the upper half-plane Im$z > 0$; (c) the real axis Im$z = 0$.

$f(z) = (1 + z)/(1 - z)$: this is a fractional linear transformation and so it takes circles to circles. (a) We take 3 points on the unit circle, $1, i, -1$. These are taken to ∞, $(1 + i)/(1 - i) = (1 + i)^2/(1 - i)(1 + i) = i$ and 0. Hence the image of the circle is the imaginary axis (circle through $0, i, \infty$). Since the unit disk is connected, it must be mapped to a connected region, i.e., to the upper or lower half-plane. Since $f(0) = 1$, the image is the upper half-plane. (b) f maps the real axis to a circle. Since $f(1) = \infty$, $f(0) = 1$ and $f(-1) = 0$, the real axis is taken to the real axis. Since the upper half-plane is connected, it is taken to the upper or lower half-plane. Since $f(i) = (1 + i)/(1 - i) = i$, the upper half-plane is taken to itself; (c) we already saw that the real axis is taken to itself.

$g(z) = \sqrt[4]{z}$: The unit disk is taken to the wedge $-\pi/4 < Arg(z) \leq \pi/4$ and $|z| \leq 1$; the upper half-plane to the wedge $-\pi/4 < Arg(z) \leq \pi/4$; the real axis to the positive reals and the ray $Arg(z) = \pi/4$.

$h(z) = \text{Arg}(z)$: the unit disk is taken to the interval $(-\pi, \pi]$, the upper half plane to $(0, \pi)$ and the real axis to $\{0\} \cup \{\pi\}$.

Problem 4. Let $u(x + iy) = \log |x + iy|$. Find a harmonic conjugate to u.

Solution. See book, example on p. 84. (You can just notice that since $\text{Log}(z) = \log |z| + i\text{Arg}(z)$, $\text{Arg}(z)$ is a harmonic conjugate to $\log |z|$).

Problem 5. Let D be a region bounded by a simple closed curve γ. Express the integral $\int_\gamma xdy - ydx$ in terms of the area of D.

Solution. Using Green's theorem we find that

$$\int_\gamma xdy - ydx = \int\int_D (\partial x/\partial x - (-\partial y/\partial y))dxdy = \int\int_D 2dxdy = 2A,$$

if A is the area of D.

Problem 6. Prove the identity $\cos(z+w) = \cos z \cos w - \sin z \sin w$ using the definition of $\cos z$ in terms of complex exponentials and the properties of complex exponentials.

Solution. See book, p. 7.

Problem 7. Let $f(z) = z(z - 1)$. Show from the definition that $f(z)$ is continuous. (Hint: prove an estimate of the form $|f(z) - f(z_0)| < C|z - z_0|$ for z near z_0).

Solution. We have

$$\begin{aligned} f(z) - f(z_0) &= z^2 - z - z_0^2 + z_0 = (z^2 - z_0^2) - (z - z_0) \\ &= (z - z_0)(z + z_0) - (z - z_0) = (z - z_0)(z + z_0 - 1). \end{aligned}$$

Hence

$$|f(z) - f(z_0)| \leq |z - z_0|\,|z + z_0 - 1|.$$

If z is near z_0 (say $|z - z_0| < 1$), then $|z| \leq |z_0| + 1$ and $|z + z_0 - 1| \leq |z| + |z_0| + 1 \leq 2|z_0| + 2$. Hence

$$|f(z) - f(z_0)| \leq 2(|z_0| + 2)|z - z_0|.$$

Now given an $\epsilon > 0$ choose $\delta > 0$, so that $\delta < 1$ and $2(|z_0| + 2)\delta < \varepsilon$. Then if $|z - z_0| < \delta$, we have that

$$|f(z) - f(z_0)| \leq 2(|z_0| + 2)|z - z_0| \leq 2(|z_0| + 2)\delta < \epsilon.$$