MATH 131B, WINTER 2004

SECOND MIDTERM SOLUTIONS.

Problem 1. State the book's version of (use next page if you need more space):
(a) Definition of convergence of a series of numbers
(b) The ratio test
(c) Banach contraction principle
(d) Definition of a complete metric space
(e) Definition of $\liminf$

See book.

Problem 2. Recall that a is a *limit point* of a sequence a_n if there is a subsequence a_{n_k} of a_n, so that $a_{n_k} \to a$. Assume that the sequence a_n is bounded.
(a) Show that $\limsup a_n$ is a limit point of a_n.
(b) Show that if a_{n_k} is a subsequence of a_n, then $\limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$
(c) Let $\mathcal{L}(a_n)$ be the set of all limit points of a_n. Use (a) and (b) to show that $\limsup a_n = \sup \mathcal{L}(a_n)$.

(a) Let $A = \limsup a_n$.

Let $\varepsilon_k = \frac{1}{k}$. Choose n_k inductively as follows. Let $n_0 = 1$. Having chosen n_{k-1}, apply a theorem in the book stating that there is an $n_k > n_{k-1}$ so that $A - \varepsilon_k \leq a_{n_k} \leq A + \varepsilon$. Then we have chosen n_k so that $A - \frac{1}{k} \leq a_{n_k} \leq A + \frac{1}{k}$. Thus $a_{n_k} \to a$ by the squeeze theorem.

(b) By definition,

$$\limsup a_{n_k} = \lim_{K\to\infty} \sup\{a_{n_k} : k > K\} \leq \lim_{K\to\infty} \sup\{a_n : n > n_K\}$$

(the last inequality is true, since if $X \subset Y$, $\sup X \leq \sup Y$). Thus if we let

$$A_N = \sup\{a_n : n > N\},$$

then we have proved:

$$\limsup a_{n_k} \leq \lim_{K\to\infty} A_{n_K}.$$

On the other hand, by definition,

$$\limsup a_n = \lim_{N\to\infty} A_N.$$

Since A_N is monotone and bounded (since a_n is bounded) it is convergent. But then any subsequence is convergent to the same limit. Thus $A_{n_K} \to \lim A_N = \limsup a_n$. It follows that

$$\limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n,$$

as claimed.

(c) Since by (a) $A = \limsup a_n$ is a limit point, $A \in \mathcal{L}(a_n)$ and thus $\sup \mathcal{L}(a_n) \geq A$. On the other hand, if $a \in \mathcal{L}(a_n)$, $a = \lim a_{n_k}$ for some subsequence $a_{n_k}$of a_n. Since $a_{n_k}$is convergent, $a = \lim a_{n_k} = \limsup a_{n_k} \leq \limsup a_n$ by part (b) . Thus $a \leq \limsup a_n$ and so $\limsup a_n \geq \sup \mathcal{L}(a_n)$.

Problem 3. Let $C[0,1]$ be the space of all continuous functions on $[0,1]$, endowed with the supremum norm $\|\cdot\|_\infty$. Let $V \subset C[0,1]$ be the subspace consisting of all functions f for which $f(0) = 3f(1)$. Is V a complete subset of $C[0,1]$? Prove that your answer is correct.

V is complete. Let f_n be a Cauchy sequence with $f_n \in V$. Then $f_n \in C[0,1]$ is also Cauchy and so converges to a function $f \in C[0,1]$, by completeness of $C[0,1]$. It suffices to show that $f \in V$. Since $f_n \in V$, $f_n(0) = 3f_n(1)$. Since $f_n \to f$ uniformly, $f_n \to f$ pointwise. Thus

$$f(0) = \lim f_n(0) = \lim 3f_n(1) = f(3)$$

and so $f \in V$.

Problem 4. State and prove the root test.

See book.