MATH 131B
2ND PRACTICE MIDTERM

Problem 1. State the book's definition of:
(a) A complete metric space
(b) $\limsup$ and $\liminf$
(c) Convergence of a series of real numbers
(d) Normed vector space; Banach space

Problem 2. Let X be a metric space with a metric ρ. Let x_n and y_n be two Cauchy sequences in X. Show that $\lim_{n\to\infty} \rho(x_n, y_n)$ exists. Note: we do *not* assume that X is complete.

Problem 3. Let ρ be the usual Eucledian metric on $\mathbb{R}$. We say that a subset $X \subset \mathbb{R}$ is *closed* if whenever $x_n \in X$ and $x_n \to x \in \mathbb{R}$, then $x \in X$. Show that a subset $X \subset \mathbb{R}$ is complete with respect to ρ if and only if it is closed.

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (3 + 0.5x + 0.1y, 4 + 0.6x).$$

Show that there is a unique point $(x_0, y_0) \in \mathbb{R}^2$ with the property that $f(x_0, y_0) = (x_0, y_0)$.

Problem 5. State and prove that Banach contraction principle.

Problem 6. Let $\|f\|_\infty$ and $\|f\|_1$ be norms on the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$, given by:

$$\begin{aligned} \|f\|_\infty &= \sup_{x\in[0,1]} |f(x)| \\ \|f\|_1 &= \int_0^1 |f(x)|dx. \end{aligned}$$

Show that the two norms are not equivalent.

Problem 7. Let $A = \limsup a_n$ and $a = \liminf a_n$. Show that $A = a$ if and only if a_n converges, and moreover that if this is the case, then $a_n \to a$.

Problem 8. State and prove the comparison test.