MATH 131B
2ND PRACTICE MIDTERM

Problem 1. State the book's definition of:
(a) A complete metric space
(b) $\limsup$ and $\liminf$
(c) Convergence of a series of real numbers
(d) Normed vector space; Banach space

Solution. See book.

Problem 2. Let X be a metric space with a metric ρ. Let x_n and y_n be two Cauchy sequences in X. Show that $\lim_{n\to\infty} \rho(x_n, y_n)$ exists. Note: we do *not* assume that X is complete.

Solution. Let $\varepsilon > 0$ be given. Choose N so that for all $n, m > N$, $\rho(x_n, x_m) < \varepsilon/2$ and $\rho(y_n, y_m) < \varepsilon/2$. This is possible because the two sequences are Cauchy.

By the triangle inequality, we have that

$$\rho(x, y) + \rho(y, z) \leq \rho(x, z)$$

for all x, y, z; this means that

$$\rho(x, y) - \rho(x, z) \leq \rho(y, z).$$

Since x, y, z are arbitrary, we can switch rhe roles of y and z and obtain that

$$|\rho(x, y) - \rho(x, z)| \leq \rho(y, z).$$

Thus:

$$\begin{aligned} |\rho(x_n, y_n) - \rho(x_m, y_m)| &= |\rho(x_n, y_n) - \rho(x_n, y_m) + \rho(y_m, x_n) - \rho(x_m, y_m)| \\ &\leq |\rho(x_n, y_n) - \rho(x_n, y_m)| + |\rho(x_n, y_m) - \rho(x_m, y_m)| \\ &\leq \rho(y_n, y_m) + \rho(x_n, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

It follows that the sequence of numbers $\{\rho(x_n, y_n)\}$ is Cauchy, and so converges.

Problem 3. Let ρ be the usual Eucledian metric on $\mathbb{R}$. We say that a subset $X \subset \mathbb{R}$ is *closed* if whenever $x_n \in X$ and $x_n \to x \in \mathbb{R}$, then $x \in X$. Show that a subset $X \subset \mathbb{R}$ is complete with respect to ρ if and only if it is closed.

Assume that X is closed. Let $x_n \in X$ be a Cauchy sequence. Then x_n is Cauchy when regarded as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_n \to x$ in $\mathbb{R}$. Since X is closed, $x \in X$ and $x_n \to x$ in X. Thus X is complete.

Assume that X is not closed. Thus there is a sequence $x_n \in X$ so that $x_n \to x$ in $\mathbb{R}$, but $x \notin X$. Since $x_n \to x$, it is Cauchy. Also, x_n does not converge in X: if $x_n \to x'$ with $x' \in X$, it would follow by uniqueness of the limit that $x = x' \in X$, but $x \notin X$. Thus x_n is a Cauchy sequence with no limit in X. Thus X is not complete.

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (3 + 0.5x + 0.1y, 4 + 0.6x).$$

Show that there is a unique point $(x_0, y_0) \in \mathbb{R}^2$ with the property that $f(x_0, y_0) = (x_0, y_0)$.

Solution. Let ρ be the metric on $\mathbb{R}^2$ given by $\rho((x, y), (x', y')) = \max(|x - x'|, |y - y'|)$. Then

$$\begin{aligned}
\rho(f(x,y), f(x',y')) &= \max(|0.5x + 0.1y - 0.5x' - 0.1y'|, |0.6x - 0.6x'|) \\
&= \max(|0.5(x - x') - 0.1(y - y')|, 0.6|x - x'|) \\
&\leq \max(0.5|x - x'| + 0.1|y - y'|, 0.6|x - x'|) \\
&\leq \max(0.5\max(|x - x'|, |y - y'|) + 0.1\max(|y - y'|, |x - x'|), 0.6|x - x'|) \\
&= \max(0.6\max(|x - x'|, |y - y'|), 0.6|x - x'|) \\
&= 0.6\max(|x - x'|, |y - y'|) \\
&= 0.6\rho((x, y), (x', y')).
\end{aligned}$$

It follows that f is a contraction. Since $\mathbb{R}^2$ is complete, we can apply the Banach contraction principle to conclude that f has a unique fixed point (x_0, y_0).

Problem 5. State and prove that Banach contraction principle.

Solution. See book.

Problem 6. Let $\|f\|_\infty$ and $\|f\|_1$ be norms on the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$, given by:

$$\begin{aligned}
\|f\|_\infty &= \sup_{x \in [0,1]} |f(x)| \\
\|f\|_1 &= \int_0^1 |f(x)| dx.
\end{aligned}$$

Show that the two norms are not equivalent.

Solution. If the two norms were equivalent, being a Cauchy sequence with respect to one of them would imply being a Cauchy sequence with respect to the other. We'll show that this is not the case.

Let f_n be given as follows. For $x \in [0, 0.5 - 1/n]$, $f_n(x) = 0$. For $x \in [0.5 + 1/n, 1]$, $f_n(x) = 1$. For $x \in (0.5 - 1/n, 0.5 + 1/n)$, $f(x) = 0.5n(x - 0.5) + 0.5$. Then $f_n \in C[0, 1]$.

Let f be given by: $f(x) = 0$ if $x \in [0, 0.5]$ and $f(x) = 1$ if $x \in (0.5, 1]$.

Then

$$\|f - f_n\|_1 = \int_0^1 |f(x) - f_n(x)| dx.$$

Since $f(x) = f_n(x)$ outside of $(0.5 - 1/n, 0.5 + 1/n)$, we get that

$$\|f - f_n\|_1 = \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} |f_n(x) - f(x)| dx.$$

Note that $|f_n(x)| \leq 1$ and $|f(x)| \leq 1$ for all x. Thus $|f_n(x) - f(x)| \leq 2$ for all x. Hence

$$\|f_n - f\|_1 \leq \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} 2dx = \frac{4}{n} \to 0.$$

It follows that $f_n \to f$ in $\| \cdot \|_1$. In particular, f_n are a Cauchy sequence for $\| \cdot \|_1$.

On the other hand, it is not hard to see that $f_n \to g$ pointwise, where $g(x) = 0$ on $[0, 0.5)$, $g(x) = 1$ on $(0.5, 1]$ and $g(0.5) = 0.5$. Thus f_n cannot be Cauchy for $\| \cdot \|_\infty$, since this would imply that $f_n \to h$ in $\|\cdot\|_\infty$ for some h and that h is continous; but since $f_n \to g$ pointwise, it follows that $h = g$, which is not possible, since g is not continuous.

Problem 7. Let $A = \limsup a_n$ and $a = \liminf a_n$. Show that $A = a$ if and only if a_n converges, and moreover that if this is the case, then $a_n \to a$.

Solution. Assume that $A = a$. The for any $\varepsilon > 0$, there is an N so that for all $n > N$, one has

$$\liminf a_n - \varepsilon < a_n < \limsup a_n + \varepsilon$$

Since $\liminf a_n = \limsup a_n = a$ in this case, we have that for all $n > N$,

$$a - \varepsilon < a_n < a + \varepsilon.$$

But then

$$|a - a_n| < \varepsilon.$$

Thus by the definition of limit, $a_n \to a$.

Conversely, suppose that $a_n \to a$. Then for any $\varepsilon > 0$, there is an N so that for all $n > N$,$|a - a_n| < \varepsilon$. Hence for $n > N$, we have

$$a - \varepsilon < a_n < a + \varepsilon.$$

It follows that for any $M > N$,

$$a - \varepsilon \leq \inf\{a_n : n > M\} \leq \sup\{a_n : n > M\} \leq a + \varepsilon.$$

Thus by the definitionof $\liminf$ and $\limsup$, we have

$$a - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq a + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\liminf a_n = \limsup a_n = a$.

Problem 8. State and prove the comparison test.

Solution. See book.