MATH 131B, WINTER 2004

FIRST MIDTERM SOLUTIONS

Problem 1. Let X be a metric space with a metric ρ. Let x_n and y_n be two Cauchy sequences in X. Show that $\lim_{n\to\infty} \rho(x_n, y_n)$ exists. Note: we do *not* assume that X is complete.

Solution. Let $\varepsilon > 0$ be given. Choose N so that for all $n, m > N$, $\rho(x_n, x_m) < \varepsilon/2$ and $\rho(y_n, y_m) < \varepsilon/2$. This is possible because the two sequences are Cauchy.

By the triangle inequality, we have that

$$\rho(x, y) + \rho(y, z) \leq \rho(x, z)$$

for all x, y, z; this means that

$$\rho(x, y) - \rho(x, z) \leq \rho(y, z).$$

Since x, y, z are arbitrary, we can switch rhe roles of y and z and obtain that

$$|\rho(x, y) - \rho(x, z)| \leq \rho(y, z).$$

Thus:

$$\begin{aligned} |\rho(x_n, y_n) - \rho(x_m, y_m)| &= |\rho(x_n, y_n) - \rho(x_n, y_m) + \rho(y_m, x_n) - \rho(x_m, y_m)| \\ &\leq |\rho(x_n, y_n) - \rho(x_n, y_m)| + |\rho(x_n, y_m) - \rho(x_m, y_m)| \\ &\leq \rho(y_n, y_m) + \rho(x_n, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

It follows that the sequence of numbers $\{\rho(x_n, y_n)\}$ is Cauchy, and so converges.

Problem 2. Show that a uniform limit of a sequence of continuous functions is continuous.

Solution. See book.

Problem 3. Let f_n be a sequence of functions on the interval $[0, +\infty)$ given by

$$f_n(x) = \frac{x}{n^2} \exp(-\frac{x}{n^2}).$$

(a) Show that f_n converge pointwise on $[0, +\infty)$ to a function f and find that function f.
(b) Show that f_n do not converge uniformly on $[0, +\infty)$.

Solution. (a) Since the exponential of a non-positive number is ≤ 1, we have that $|f_n(x)| \leq |x|/n^2$. For x fixed, $x/n^2 \to 0$ as $n \to \infty$, so $f_n(x)$ converge to the zero function pointwise.

(b) Since $f_n(n^2) = \exp(-1)$, it follows that $\|f_n\|_\infty \geq 1/e$. Thus f_n cannot converge to zero uniformly, as this would imply that $\|f_n\|_\infty \to 0$.

Problem 4. Let f_n be a sequence of functions on the interval $[0,1]$, so that f_n converges uniformly on $[0,1]$ to a function f. Let x_n be a sequence of points in $[0,1]$ so that $x_n \to 0$. Show that $f_n(x_n) \to f(0)$.

Solution. Because of uniform convergence, we have that $\|f - f_n\|_\infty \to 0$. By definition of the sup norm, for any y,

$$|f_n(y) - f(y)| \leq \|f - f_n\|_\infty.$$

Applying this to $y = x_n$ gives

$$|f_n(x_n) - f(x_n)| \leq \|f - f_n\|_\infty \to 0,$$

so that $|f_n(x_n) - f(x_n)| \to 0$, so that $f_n(x_n) - f(x_n) \to 0$. Since f is continuous (beign a uniform limit of continuous functions), $f(x_n) \to f(0)$. Since the sum of two convergent sequences is convergent tot he sum of the limits, we deduce that

$$f_n(x_n) = (f_n(x_n) - f(x_n)) + f(x_n) \to 0 + f(0).$$