MATH 131B
1ST PRACTICE MIDTERM

Problem 1. State the book's definition of:
(a) Uniform convergence of a sequence of function
(b) A metric
(c) Pointwise convergence of a sequence of functions
(d) A continuously differentiable function of two variables
(e) A continuous function of two variables
(f) The supremum norm $\|f\|_\infty$

Problem 2. Prove that a uniform limit of a sequence of continuous functions is also continuous.

Problem 3. Let f_n be a sequence of functions that converges uniformly on $[a, b]$ to a function f. Show that

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Give an example which shows that the conclusion does not hold if we only assume pointwise convergence.

Problem 4. Let $C^1[0, 1]$ be the space of functions on $[0, 1]$ which have a continuous derivative. Let

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

(a) Show that $\|f\|$ is a norm
(b) Show that $C^1[0, 1]$ is complete in this norm.

Problem 5. Endow $\mathbb{R}$ with the following metric ρ: $\rho(x, y) = 1$ if $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. (a) Show that ρ is a metric. (b) Show that a sequence $x_n \in \mathbb{R}$ is Cauchy for this metric if and only if it is eventually constant; i.e., for some N, $x_n = x_m$ whenever $n, m > N$. (c) Show that $\mathbb{R}$ is complete with respect to this metric.

Problem 6. Show that $f_n = \frac{n^2 x}{1-n^4x^2}$ converges pointwise to 0 on $[0, 1]$, but does not converge uniformly.

Problem 7. Let $f : [0, 1] \to [0, 1]$ be a continuously differentiable function. Suppose that $|f'(x)| < 1$ for all $x \in [0, 1]$. Show that f is a "uniform contraction": there is a constant $0 \leq C < 1$ so that for all $x, y \in [0, 1]$, $|f(x) - f(y)| \leq C|x - y|$.