

MATH 131B
1ST PRACTICE MIDTERM
SOLUTIONS

Problem 1. State the book's definition of:

- (a) Uniform convergence of a sequence of function
- (b) A metric
- (c) Pointwise convergence of a sequence of functions
- (d) A continuously differentiable function of two variables
- (e) A continuous function of two variables
- (f) The supremum norm $\|f\|_\infty$

Solution. See the following pages: (a), p. 167; (b), 196; (c), 164; (d), 155; (e) 152; (f) 175.

Problem 2. Prove that a uniform limit of a sequence of continuous functions is also continuous.

Solution. See proof of Theorem 5.2.1 on pages 169-170.

Problem 3. Let f_n be a sequence of functions that converges uniformly on $[a, b]$ to a function f . Show that

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx.$$

Give an example which shows that the conclusion does not hold if we only assume pointwise convergence.

Solution. For the first part, see proof of Theorem 5.2.2 on p. 170. For the second part, see Example 4 on page 166.

Problem 4. Let $C^1[0, 1]$ be the space of functions on $[0, 1]$ which have a continuous derivative. Let

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

- (a) Show that $\|f\|$ is a norm
- (b) Show that $C^1[0, 1]$ is complete in this norm.

Solution. (a) We must prove: (i) $\|f\| = 0$ iff $f = 0$; (ii) $\|\alpha f\| = |\alpha| \|f\|$ and (iii) $\|f + g\| \leq \|f\| + \|g\|$.

If $\|f\| = 0$ then for sure $\|f\|_\infty = 0$. Thus $f = 0$ since $\|f\|_\infty$ is a norm. Conversely, if $f = 0$, then $\|f\| = 0$.

Since $\|f\| = \sup_x |f(x)| + \sup_y |f'(y)|$, we have that $\|\alpha f\| = |\alpha| \|f\|$.

Similarly,

$$\begin{aligned} \|f + g\| &= \sup_x |f(x) + g(x)| + \sup_y |f'(y) + g'(y)| \\ &\leq \sup_x |f(x)| + \sup_x |g(x)| + \sup_y |f'(y)| + \sup_y |g'(y)| \\ &= \|f\| + \|g\|. \end{aligned}$$

(b) We must prove that if f_n is a Cauchy sequence, then f_n converges in this norm to a continuously differentiable function.

Note that $\|f\| \geq \|f\|_\infty$ and $\|f\| \geq \|f'\|_\infty$. Thus if f_n is a Cauchy sequence with respect to the norm $\|\cdot\|$, then f_n' is a Cauchy sequence for the uniform norm $\|\cdot\|_\infty$. Since the space of continuous functions is complete in the uniform norm, it follows that the sequence f_n' converges in uniform norm to a function $g \in C[0, 1]$. Moreover, since f_n form a Cauchy sequence in the uniform norm, then $f_n(0)$ form a Cauchy sequence and thus converge.

We now apply Theorem 5.2.3 with $x_0 = 0$. The conclusion is that the sequence f_n converges uniformly to a function f , which is continuously differentiable and whose derivative is g . Note that we thus have that $f \in C^1[0, 1]$.

Since $f_n \rightarrow f$ uniformly, it follows that $\|f - f_n\|_\infty \rightarrow 0$, by Proposition 5.3.2. Similarly, since $f_n' \rightarrow g = f'$ uniformly, $\|f' - f_n'\|_\infty \rightarrow 0$. Thus $\|f - f_n\| = \|f - f_n\|_\infty + \|f' - f_n'\|_\infty \rightarrow 0$, so that $f \rightarrow f_n$ in the norm $\|\cdot\|$.

Problem 5. Endow \mathbb{R} with the following metric ρ : $\rho(x, y) = 1$ if $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. (a) Show that ρ is a metric. (b) Show that a sequence $x_n \in \mathbb{R}$ is Cauchy for this metric if and only if it is eventually constant; i.e., for some N , $x_n = x_m$ whenever $n, m > N$. (c) Show that \mathbb{R} is complete with respect to this metric.

Solution. (a) It is trivial to see that $\rho(x, y) = \rho(y, x)$ and that $\rho(x, y) = 0$ iff $x = y$. For the triangle inequality, we must show that $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. The left hand side is 1 unless $x = z$. The right hand side is either 0, 1 or 2, and so it is at least as big as the left hand in the last two cases. If the right hand side is 0, then $\rho(x, y) = \rho(y, z) = 0$ so $x = y = z$ and the triangle inequality still holds. (b) Let $\varepsilon = 1/2$. Then there is an N so that for all $n, m > N$, $\rho(x_n, x_m) < 1/2$. But this means that $x_n = x_m$ for all $n, m > N$. (c) Let x_n be a Cauchy sequence. From part (b) we know that there is an N so that $x_n = x_m$ for all $n, m > N$. Fix $m > N$ and let $x = x_m$. Then $x_n \rightarrow x$. Indeed, for any $\varepsilon > 0$, we have that for all $n > N$, $\rho(x, x_n) = \rho(x_m, x_n) = 0$, since $x_n = x_m$.

Problem 6. Show that $f_n = \frac{n^2 x}{1 - n^4 x^2}$ converges pointwise to 0 on $[0, 1]$, but does not converge uniformly.

Solution. For each fixed x ,

$$\lim_{n \rightarrow \infty} \frac{n^2 x}{1 - n^4 x^2} = \lim_{n \rightarrow \infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4} - x^2} = 0.$$

Thus the sequence converges pointwise.

On the other hand, since $f_n(\frac{1}{n^2}) = \frac{1}{2}$, it follows that $\|f_n\|_\infty \geq 1/2$, so that $\|f_n\|_\infty$ does not converge to zero. Thus f_n do not converge to zero uniformly.

Problem 7. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuously differentiable function. Suppose that $|f'(x)| < 1$ for all $x \in [0, 1]$. Show that f is a “uniform contraction”: there is a constant $0 \leq C < 1$ so that for all $x, y \in [0, 1]$, $|f(x) - f(y)| \leq C|x - y|$.

Solution. Since $f'(x)$ is continuous, so is $|f'(x)|$. Hence $|f'(x)|$ attains its maximum at some point $x_0 \in [0, 1]$. It follows that if we set $C = |f'(x_0)|$, then $C < 1$ and $|f'(z)| \leq |f'(x_0)| = C$ for all z in $[0, 1]$.

By applying the Mean Value Theorem, we obtain that for all $x, y \in [0, 1]$ there is a point $z \in [x, y]$, so that $f(x) - f(y) = f'(z)(x - y)$. We conclude that

$$|f(x) - f(y)| = |f'(z)||x - y| \leq C|x - y|,$$

where C as above is a constant strictly less than 1.