Modular symbols and Milnor K_2

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1 Symbols

Recall that Milnor's second K-group for a commutative ring R with unit is defined by

$$K_2^M(R) = \frac{R^\times \otimes_{\mathbf{Z}} R^\times}{\langle a \otimes (1-a) \mid a, 1-a \in R^\times \rangle}.$$

Let $N \geq 1$ be prime to 6. Let $F = \mathbf{Q}(\mu_N)$ and $\Delta = \mathrm{Gal}(F/\mathbf{Q})$, which we may identify with $(\mathbf{Z}/N\mathbf{Z})^\times$ via the Teichmüller character. We let $\sigma_j \in \Delta$ denote the element corresponding to $j \in \mathbf{Z}$ prime to N. For any algebraic extension E/F, let S denote the set of primes of F dividing N. We consider the ring $\mathcal{O}_{F,S}$ of S-integers of F. Let $\mathbf{Z}' = \mathbf{Z}[1/2, 1/3]$. We denote the element of $K_2^M(\mathcal{O}_{F,S})^+ \otimes_{\mathbf{Z}} \mathbf{Z}'$ represented by $a \otimes b$ for $a, b \in \mathcal{O}_{F,S}^\times$ by (a, b). Here, "+" denotes the fixed part under complex conjugation.

We let ζ_N denote a primitive Nth root of unity. Note that $(\zeta_N, 1-\zeta_N) = 0$, from which we easily conclude $(\zeta_N, \zeta_N) = 0$. Using this, we see that $(\zeta_N^a, 1-\zeta_N^b) = 0$ for any $a, b \in \mathbf{Z}$ not divisible by N, since the symbol then lies in the minus part of Milnor K_2 (up to 2-torsion). It is also not hard to check that

$$(1-\zeta_N^a, 1-\zeta_N^b) + (1-\zeta_N^b, 1-\zeta_N^a) = 0 \tag{1}$$

(see [McS, Lemma 3.2]). If $a-b$ is not divisible by N as well, then the identity

$$1-\zeta_N^{a-b} + \zeta_N^{a-b}(1-\zeta_N^b) = 1-\zeta_N^a,$$

implies

$$(1-\zeta_N^a, 1-\zeta_N^b) + (1-\zeta_N^{a-b}, 1-\zeta_N^a) + (1-\zeta_N^b, 1-\zeta_N^{a-b}) = 0. \tag{2}$$

Note also that, since $1-\zeta_N^a = -\zeta_N^a(1-\zeta_N^{-a})$, we have

$$(1-\zeta_N^a, 1-\zeta_N^b) = (1-\zeta_N^{-a}, 1-\zeta_N^b).$$

Furthermore, we have

$$\sigma_j(1-\zeta_N^a, 1-\zeta_N^b) = (1-\zeta_N^{ja}, 1-\zeta_N^{jb}).$$

Let D denote the group $(\mathbf{Z}/N\mathbf{Z})^\times$ of diamond operators on $H_1(X_1(N), C_1(N); \mathbf{Z}')$, where $C_1(N)$ denotes the cusps on the modular curve $X_1(N)$. For $u, v \in \mathbf{Z}/N\mathbf{Z}$ with $(u, v) = \mathbf{Z}/N\mathbf{Z}$, we define

$$[u:v] = \left\{ \frac{a}{Nc}, \frac{b}{Nd} \right\},$$

the class of the geodesic from a/Nc to b/Nd, where $a, b, c, d \in \mathbf{Z}$ satisfy $ad - bc = 1$, $u = a \pmod N$, and $v = b \pmod N$. These symbols are exactly the usual Manin symbols with the Atkin-Lehner involution applied. We shall refer to them simply as Manin symbols. Now $H_1(X_1(N), C_1(N), \mathbf{Z}')$ has a presentation a $\mathbf{Z}'[D]$-module with generators $[u:v]$, subject to the relations $[-u:-v] = [u:v]$,

$$[u:v] + [-v:u] = 0 \quad \text{and} \quad [u:v] + [u-v:u] + [-v:u-v] = 0, \tag{3}$$

along with

$$\langle j \rangle [u:v] = [j^{-1}u : j^{-1}v]$$

Here we use $\langle j \rangle$ to denote the jth diamond operator.

Note that the projection of $[u:v]$ to $H_1(X_1(N), C_1(N); \mathbf{Z}')^+$ is invariant under the transformation $u \mapsto -u$. Comparing (1) and (2) to (3), we then see that

$$[u:v] \mapsto (1-\zeta_N^u, 1-\zeta_N^v)$$

for $u, v \neq 0$ and $[0:v], [u:0] \mapsto 0$ define a homomorphism

$$\Phi\colon H_1(X_1(N), C_1(N); \mathbf{Z}') \to K_2^M(\mathcal{O}_{F,S})^+ \otimes_{\mathbf{Z}} \mathbf{Z}'$$

factoring through $H_1(X_1(N), C_1(N); \mathbf{Z}')^+$. We remark that for nonzero u and v, we have

$$\Phi(\langle j \rangle [u:v]) = \sigma_j^{-1}(1-\zeta_N^u, 1-\zeta_N^v).$$

2 Hecke operators

Let us continue to assume that both u and v are both nonzero in $\mathbf{Z}/N\mathbf{Z}$. Note the identity:

$$(1-\zeta_N^u)(1-\zeta_N^{u+v})(1-\zeta_N^v) + \zeta_N^u(1-\zeta_N^u)(1-\zeta_N^{2v}) = (1-\zeta_N^v)(1-\zeta_N^{2u}).$$

Thus, if $u+v \neq 0$, we have

$$\left(\frac{(1-\zeta_N^u)(1-\zeta_N^{u+v})}{1-\zeta_N^{2u}}, \frac{(1-\zeta_N^u)(1-\zeta_N^{2v})}{(1-\zeta_N^v)(1-\zeta_N^{2u})}\right) = 0.$$

It follows that

$$\begin{aligned}&(1-\zeta_N^u, 1-\zeta_N^{2v}) + (1-\zeta_N^{2u}, 1-\zeta_N^v) + (1-\zeta_N^{2u}, 1-\zeta_N^{u+v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^{2v}) \\ &\quad = (1-\zeta_N^u, 1-\zeta_N^v) + (1-\zeta_N^{2u}, 1-\zeta_N^{2v}) + (1-\zeta_N^u, 1-\zeta_N^{u+v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^v).\end{aligned}$$

Applying (2) to the last two terms, we obtain

$$\begin{aligned}&(1-\zeta_N^u, 1-\zeta_N^{2v}) + (1-\zeta_N^{2u}, 1-\zeta_N^v) + (1-\zeta_N^{2u}, 1-\zeta_N^{u+v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^{2v}) \\ &\qquad = (2+\sigma_2)(1-\zeta_N^u, 1-\zeta_N^v) \quad (4)\end{aligned}$$

Consider the Atkin-Lehner involution

$$w = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We compute first the action of dual operators $T_l^* = wT_lw^{-1}$. The action of T_l then easily follows from [DS, Theorem 5.5.3], which states that

$$T_l^* = \langle l \rangle^{-1} T_l$$

for l prime to N. Note that T_2^* acts on Manin symbols as [Me, Proposition 20]:

$$T_2^*[u:v] = [u:2v] + [2u:v] + [2u:u+v] + [u+v:2v],$$

Hence, (4) implies

$$\Phi(T_2[u:v]) = (1+2\sigma_2^{-1})\Phi([u:v]).$$

(Note: since $[u:-u]=0$, this holds for all $u, v \neq 0$.)

Next, consider the identity

$$(1-\zeta_N^v)(1-\zeta_N^{u+v})(1-\zeta_N^u)(1-\zeta_N^{u-v}) + \zeta_N^{u-v}(1-\zeta_N^{3v})(1-\zeta_N^u) = (1-\zeta_N^v)(1-\zeta_N^{3u}),$$

which yields

$$\left(\frac{(1-\zeta_N^{u-v})(1-\zeta_N^u)(1-\zeta_N^{u+v})}{1-\zeta_N^{3u}}, \frac{(1-\zeta_N^u)(1-\zeta_N^{3v})}{(1-\zeta_N^v)(1-\zeta_N^{3u})}\right) = 0.$$

It follows that

$$
\begin{aligned}
&(1-\zeta_N^u, 1-\zeta_N^{3v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^{3v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{u+v}) \\
&\quad + (1-\zeta_N^{u-v}, 1-\zeta_N^{3v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{u-v}) = (1-\zeta_N^{u}, 1-\zeta_N^{v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{3v}) \\
&\quad + (1-\zeta_N^{u}, 1-\zeta_N^{u+v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^{v}) + (1-\zeta_N^{u}, 1-\zeta_N^{u-v}) + (1-\zeta_N^{u-v}, 1-\zeta_N^{v}).
\end{aligned}
$$

By (4), the last four terms reduce to $2(1-\zeta_N^u, 1-\zeta_N^v)$. Similarly, we have

$$(1-\zeta_N^{3u}, 1-\zeta_N^{u-v}) = (1-\zeta_N^{3u}, 1-\zeta_N^{2u+v}) + (1-\zeta_N^{2u+v}, 1-\zeta_N^{u-v})$$

and

$$(1-\zeta_N^{u-v}, 1-\zeta_N^{3v}) = (1-\zeta_N^{u+2v}, 1-\zeta_N^{3v}) + (1-\zeta_N^{u-v}, 1-\zeta_N^{u+2v}),$$

and the sum of the last term in each of these is $(1-\zeta_N^{2u+v}, 1-\zeta_N^{u+2v})$. Therefore, we have

$$
\begin{aligned}
&(1-\zeta_N^u, 1-\zeta_N^{3v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{v}) + (1-\zeta_N^{u+v}, 1-\zeta_N^{3v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{u+v}) \\
&\quad + (1-\zeta_N^{2u+v}, 1-\zeta_N^{u+2v}) + (1-\zeta_N^{3u}, 1-\zeta_N^{2u+v}) + (1-\zeta_N^{u+2v}, 1-\zeta_N^{3v}) \\
&\quad = (3+\sigma_3)(1-\zeta_N^u, 1-\zeta_N^v). \quad (5)
\end{aligned}
$$

Now, T_3^* acts as

$$
\begin{aligned}
T_3^*[u:v] &= [u:3v] + [3u:v] + [3u:u+v] + [u+v:3v] \\
&\quad + [3u:u+2v] + [2u+v:3v] + [2u+v:u+2v],
\end{aligned}
$$

so (5) implies

$$\Phi(T_3[u:v]) = (1+3\sigma_3^{-1})\Phi([u:v]).$$

References

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Acknowledgments. (Added in 2013.) I typed this note for Glenn Stevens a couple of days after he informed me that Cecelia Busuioc had proven something similar. Busuioc subsequently published her work, which contains what is essentially this computation and more. I hope the reader will enjoy the simplicity of this computation, which involves the relations among cup products that Bill McCallum and I had previously discovered.