A proof of a duality result of Fukaya and Kato

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1 The theorem

We suppose that Λ is a (p-)adic ring for a prime p, in the sense of [FK, Section 1.4]. That is, letting J denote the radical of Λ, one has that Λ/J^n for any positive integer n is finite of order a power of p and Λ is isomorphic to the inverse limit of the Λ/J^n. We give Λ the resulting profinite topology. Recall that, in particular, Λ/J is isomorphic to a product of matrix algebras over finite fields of characteristic p.

Let F be a finite extension of $\mathbf{Q}_\ell$ for some prime number ℓ, and let G denote its absolute Galois group. Suppose that T is a projective finitely generated (left) Λ-module with a commuting action of G. We denote the complex of continuous G-cochains with values in T by $C(G,T)$. The resulting object in the derived category is denoted $R\Gamma(G,T)$, and its ith cohomology group is denoted $H^i(G,T)$. These are all naturally Λ-modules.

Let Λ° denote the opposite ring of Λ. Set

$$T^* = \mathrm{Hom}_\Lambda(T,\Lambda).$$

Then T^* is naturally a $\Lambda^\circ[G]$-module, using the standard G-action considering Λ as a trivial G-module and right multiplication on T^* by elements of Λ to give the Λ°-action. Moreover, the Pontryagin dual

$$T^\vee = \mathrm{Hom}_{\mathrm{cts}}(T, \mathbf{Q}_p/\mathbf{Z}_p)$$

is also naturally a discrete right $\Lambda^\circ[G]$-module, with the G-action defined similarly and the Λ°-action arising in the standard way from the Λ-action on T.

We wish to prove the following result, stated by Fukaya and Kato but with the proof omitted. Note that they state several other similar results. This proof serves as a prototype for the others.

Theorem 1.1 (Fukaya-Kato)**.** *We have canonical isomorphisms in the derived category of complexes of left Λ-modules:*

$$\begin{aligned} \Psi_\Lambda(F,T)\colon \quad & R\Gamma(G,T) \xrightarrow{\sim} R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)),\Lambda)[-2], \\ \Phi_\Lambda(F,T)\colon \quad & R\Gamma(G,T) \xrightarrow{\sim} R\mathrm{Hom}_{\mathbf{Z}}(R\Gamma(G,T^\vee(1)),\mathbf{Q}_p/\mathbf{Z}_p)[-2] \end{aligned}$$

Here, $R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)),\Lambda)$ is given a left action via left multiplication on Λ and $R\mathrm{Hom}_{\mathbf{Z}}(R\Gamma(G,T^\vee(1)),\mathbf{Q}_p/\mathbf{Z}_p)$ is given a left action in the obvious fashion. The maps in question are induced by cup products

$$\begin{gathered} C(G,T)\otimes_{\mathbf{Z}} C(G,T^*(1)) \to \tau^{\geq 2}C(G,\Lambda(1)) \\ C(G,T^\vee(1))\otimes_\Lambda C(G,T) \to \tau^{\geq 2}C(G,\mathbf{Q}_p/\mathbf{Z}_p(1)), \end{gathered}$$

where we have used the canonical evaluation maps and then passed to a truncation of the complex on the right, given as

$$\tau^{\geq 2}C^i(G,A) \cong \begin{cases} 0 & i \leq 1 \\ C^2(G,A)/B^2(G,A) & i = 2 \\ C^i(G,A) & i \geq 3, \end{cases}$$

where $B^2(G,A)$ denotes the group of continuous 2-coboundaries and with the obvious maps, for A any continuous G-module. By local class field theory, this complex is then quasi-isomorphic to a complex concentrated in degree 2 and isomorphic in that degree to Λ and $\mathbf{Q}_p/\mathbf{Z}_p$ in the respective cases. The maps $\Psi_\Lambda(F,T)$ and $\Phi_\Lambda(F,T)$ are then given by taking adjoints (on opposite sides) and passing to the derived category. One sees easily that these are Λ-modules, noting that the tensor product is a Λ-bimodule in the first case and is taken over Λ in the second.

We also remark that the Pontryagin dual of a discrete module is in fact its discrete dual, and that since $\mathbf{Q}_p/\mathbf{Z}_p$ is injective as a $\mathbf{Z}$-module, $R\mathrm{Hom}_{\mathbf{Z}}$ is simply $\mathrm{Hom}_{\mathbf{Z}}$ in this instance. As a consequence, this tells us that $\Phi_\Lambda(F,T)$ is an isomorphism in the derived category of complexes of abelian groups if and only if the pairings

$$H^i(G,T)\times H^{2-i}(G,T^\vee(1)) \to \mathbf{Q}_p/\mathbf{Z}_p$$

on cohomology groups are perfect. That $\Phi_\Lambda(F,T)$ is an isomorphism is therefore a consequence of local class field theory, which is to say, Tate duality (using first the finite module T/J^n, and then taking limits), and that it is a Λ-module homomorphism is immediate. The proof that $\Psi_\Lambda(F,T)$ is an isomorphism is considerably more involved and will take the rest of this note.

2 The proof

The proof that $\Psi_\Lambda(F,T)$ is an isomorphism will consist of three steps. The first is to prove it with Λ for which $J=0$, i.e., for a product of matrix rings over finite extensions of $\mathbf{F}_p$. The second is to prove it inductively for the ring Λ/J^n for any $n \geq 1$. The third is to pass to the inverse limit.

In the case that $J=0$, we claim that we actually obtain perfect pairings, which as remarked above we need only verify on the level of cohomology. Note that for $\Lambda = \mathbf{F}_p$, then this is simply a consequence of the fact that $\Phi_{\mathbf{F}_p}(F,T)$ is an isomorphism. Suppose then that $\Lambda = \mathbf{F}_q$ for q a power of p. Let $\mathrm{Tr}\colon \mathbf{F}_q \to \mathbf{F}_p$ denote the trace map, and let

$$\mathrm{Tr}'\colon \mathrm{Hom}_{\mathbf{F}_q}(T,\mathbf{F}_q) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{F}_p}(T,\mathbf{F}_p)$$

denote the resulting induced isomorphism of $\mathbf{F}_p[G]$-modules: $\mathrm{Tr}'(f)(m) = \mathrm{Tr}(f(m))$ for any $\mathbf{F}_q$-linear homomorphism $f\colon T \to \mathbf{F}_q$ and $m \in T$. For any $i \in \mathbf{Z}$, we have a commutative diagram

$$\begin{array}{ccccc}
H^i(G,T) & \times & H^{2-i}(G,\mathrm{Hom}_{\mathbf{F}_q}(T,\mathbf{F}_q)(1)) & \longrightarrow & \mathbf{F}_q \\
\Big\| & & \cong\Big\downarrow \mathrm{Tr}' & & \Big\downarrow \mathrm{Tr} \\
H^i(G,T) & \times & H^{2-i}(G,\mathrm{Hom}_{\mathbf{F}_p}(T,\mathbf{F}_p)(1)) & \longrightarrow & \mathbf{F}_p.
\end{array}$$

From the nondegeneracy of the resulting lower pairing of $\mathbf{F}_p$-vector spaces, one sees immediately the nondegeneracy of the upper pairing of $\mathbf{F}_q$-vector spaces (counting dimensions, for instance).

Next, we consider the case that $\Lambda = M_n(\mathbf{F}_q)$ for some q as before and $n \geq 1$. Recall that Morita equivalence tells us that the category of left $M_n(\mathbf{F}_q)$-modules is equivalent to the category of $\mathbf{F}_q$-vector spaces, where an $\mathbf{F}_q$-vector space T corresponds to the $M_n(\mathbf{F}_q)$-module T^n, with the standard action of left multiplication of a column vector by a matrix. If T^n is a left $M_n(\mathbf{F}_q)[G]$-module, we note that G must act diagonally on T^n for its action to commute with the matrix action. We have an induced isomorphism of $\mathbf{F}_q[G]$-modules

$$Q\colon \mathrm{Hom}_{\mathbf{F}_q}(T,\mathbf{F}_q^n) \xrightarrow{\sim} \mathrm{Hom}_{M_n(\mathbf{F}_q)}(T^n, M_n(\mathbf{F}_q))$$

that takes $f\colon T \to \mathbf{F}_q^n$ to the homomorphism taking $(t_1,\ldots,t_n) \in T^n$ to the matrix with ith row $f(t_i)$. Again, for any $i \in \mathbf{Z}$, we have a commutative diagram

$$\begin{array}{ccccc}
H^i(G,T^n) & \times & H^{2-i}(G,\mathrm{Hom}_{M_n(\mathbf{F}_q)}(T^n,M_n(\mathbf{F}_q))(1)) & \longrightarrow & M_n(\mathbf{F}_q) \\
\Big\| & & \cong\Big\downarrow Q^{-1} & & \Big\| \\
H^i(G,T)^n & \times & H^{2-i}(G,\mathrm{Hom}_{\mathbf{F}_q}(T,\mathbf{F}_q)(1))^n & \longrightarrow & M_n(\mathbf{F}_q).
\end{array}$$

Here, the (a, b)th entry of the cup product in the bottom row is exactly the cup product of the ath entry of $H^i(G, T)^n$ and the bth entry of $H^{2-i}(G, \mathrm{Hom}_{\mathbf{F}_q}(T, \mathbf{F}_q)(1))^n$, and each of these individual pairings to $\mathbf{F}_q$ is perfect. One sees from this and dimension counting that the induced pairing of left/right $M_n(\mathbf{F}_q)$-modules on the top row is perfect in the sense that it induces isomorphisms

$$H^i(G, T^n) \xrightarrow{\sim} \mathrm{Hom}_{M_n(\mathbf{F}_q)^\circ}(H^{2-i}(G, \mathrm{Hom}_{M_n(\mathbf{F}_q)}(T^n, M_n(\mathbf{F}_q))(1)), M_n(\mathbf{F}_q))$$

and

$$H^{2-i}(G, \mathrm{Hom}_{M_n(\mathbf{F}_q)}(T^n, M_n(\mathbf{F}_q))(1)) \xrightarrow{\sim} \mathrm{Hom}_{M_n(\mathbf{F}_q)}(H^i(G, T^n), M_n(\mathbf{F}_q))$$

of left and right Λ-modules, respectively.

Finally, we remark that the general case of $J = 0$ follows easily from this, by applying idempotents to work one matrix algebra at a time, and then summing them together again. That is, we have proven that $\Psi_{\Lambda/J}(F, T/J)$ is an isomorphism for general Λ and T.

Remark. In a note to the author, Meng Fai Lim gave a different and quite short proof of the fact that $\Psi_{\Lambda/J}(F, T/J)$ is an isomorphism, using the fact that Λ/J is self-injective.

We next recall the key results of Fukaya-Kato (for the group G at hand) [FK, Proposition 1.6.5] and [FK, Lemma 1.6.8] that we shall need.

Theorem 2.1 (Fukaya-Kato)**.** *Suppose that Λ' is a (p-)adic ring and we are given a continuous ring homomorphism $\Lambda \to \Lambda'$. One has a bounded complex L of finitely generated projective Λ-modules and a homomorphism of complexes of Λ-modules $L \to C(G, T)$ such that the induced map*

$$M \otimes_\Lambda L \to C(G, M \otimes_\Lambda T)$$

is a quasi-isomorphism for any finitely generated projective left Λ'-module M with a trivial G-action. Moreover, for any such M, the natural map on complexes induces an isomorphism

$$M \otimes^L_\Lambda R\Gamma(G, T) \xrightarrow{\sim} R\Gamma(G, M \otimes_\Lambda T).$$

We now proceed to the inductive step. For any $n \geq 1$, we have

$$T^*/J^nT^* \cong \mathrm{Hom}_\Lambda(T/J^n, \Lambda/J^n),$$

as T is projective. Therefore, we have natural maps $T^* \otimes_\Lambda T/J^n \to \Lambda/J^n$ for every n. Taking cup products and adjoints, these provide a commutative diagram of exact triangles in the derived category of Λ-modules

$$\begin{array}{ccc} R\Gamma(G, J^nT/J^{n+1}T) & \longrightarrow & R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)), J^n/J^{n+1})[-2] \\ \downarrow & & \downarrow \\ R\Gamma(G, T/J^{n+1}T) & \longrightarrow & R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)), \Lambda/J^{n+1})[-2] \\ \downarrow & & \downarrow \\ R\Gamma(G, T/J^nT) & \longrightarrow & R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)), \Lambda/J^n)[-2]. \end{array} \tag{1}$$

Suppose that we know that $\Psi_{\Lambda/J^n}(F, T/J^nT)$ is an isomorphism for some $n \geq 1$. We apply Theorem 2.1 to find a bounded complex P of finitely generated projective Λ°-modules such that both $P \to C(G, T^*(1))$ and the induced map $P/J^n \to C(G, T^*/J^nT^*(1))$ are quasi-isomorphisms. We have that

$$\mathrm{Hom}_{\Lambda^\circ}(P, \Lambda/J^n) \cong \mathrm{Hom}_{(\Lambda/J^n)^\circ}(P/J^n, \Lambda/J^n)$$

and, as P consists of projectives, these compute the derived Hom functors. Replacing P with the chain complexes in question, we have

$$R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)), \Lambda/J^n) \cong R\mathrm{Hom}_{(\Lambda/J^n)^\circ}(R\Gamma(G, \mathrm{Hom}_\Lambda(T/J^n, \Lambda/J^n)(1)), \Lambda/J^n).$$

By induction, the bottom horizontal map in (1) is now seen to be an isomorphism. We will be done with the induction if we can prove the top horizonal map is.

Let $T' = T/JT$ and $\Lambda' = \Lambda/J$. Applying an identical argument to the above (with $n = 1$), we have

$$R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G,T^*(1)), J^n/J^{n+1}) \cong R\mathrm{Hom}_{(\Lambda')^\circ}(R\Gamma(G, \mathrm{Hom}_{\Lambda'}(T', \Lambda')(1)), J^n/J^{n+1}).$$

Since all finite Λ'-modules (and in particular J^n/J^{n+1}) are projective, the latter Λ'-module may be rewritten as

$$J^n/J^{n+1} \otimes_{\Lambda'} R\mathrm{Hom}_{(\Lambda')^\circ}(R\Gamma(G, \mathrm{Hom}_\Lambda(T', \Lambda')(1)), \Lambda').$$

Applying Theorem 2.1 to Λ'-module in the upper-left hand corner of (1), we may then rewrite the top row of (1) as

$$J^n/J^{n+1} \otimes_{\Lambda'} R\Gamma(G, T') \to J^n/J^{n+1} \otimes_{\Lambda'} R\mathrm{Hom}_{(\Lambda')^\circ}(R\Gamma(G, \mathrm{Hom}_\Lambda(T', \Lambda')(1)), \Lambda')[-2],$$

This map is just $\mathrm{id} \otimes_{\Lambda'} \Psi_{\Lambda'}(F, T')$, and so that it is an isomorphism follows from the case $n = 1$ already proven.

Now, we must pass to the limit. For this, we only need note that, as a consequence of Theorem 2.1, we have isomorphisms

$$\Lambda/J^n \otimes^{L}_{\Lambda} R\Gamma(G, T) \xrightarrow{\sim} R\Gamma(G, T/J^nT)$$

and

$$\Lambda/J^n \otimes^{L}_{\Lambda} R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G, T^*(1)), \Lambda) \xrightarrow{\sim} R\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(G, T^*(1)), \Lambda/J^n).$$

(For the latter, we can replace $R\Gamma(G, T^*(1))$ with a complex of projective Λ°-modules to see the isomorphism.) The result now follows by taking an inverse limit over n.

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