Eisenstein cocycles in motivic cohomology

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The maps

Goals

1. To give a new construction of explicit maps of Busuioc and S. for $N \geq 1$:

$$\Pi_N : H_1(X_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N])^+, \quad [c:d] \mapsto \{1-\zeta_N^c, 1-\zeta_N^d\}.$$

   taking (projections of) Manin symbols to Steinberg symbols.
   Here, $+$ denotes the part fixed by complex conjugation after inverting 2.

2. To verify that Π_N is Eiseinstein, i.e., factors through the quotient of homology by an Eisenstein ideal I in the weight 2 Hecke algebra.

Details

- The symbols in question lie in a homology group relative to cusps $C_1^\circ(N)$ not over $\infty \in X_0(N)$ and the second K-group of $\mathbb{Z}[\mu_N, \frac{1}{N}]$. We define a map Π_N° on these groups and restrict.
- The Manin symbols are classes of geodesics $[c:d] = \{\frac{a}{c} \to \frac{b}{d}\}$ between cusps, where $ad - bc = 1$. They depend only on (nonzero) c, d modulo N.
- The Steinberg symbols $\{1-\zeta_N^c, 1-\zeta_N^d\}$ are of cyclotomic N-units. Here, $\zeta_N = e^{2\pi i/N}$, viewing $\overline{\mathbb{Q}} \subset \mathbb{C}$.
- The Eisenstein ideal I is generated by $T_\ell - 1 - \ell\langle\ell\rangle$ for primes ℓ, where we take $\langle\ell\rangle = 0$ if $\ell \nmid N$. The action is via dual correspondences on $X_1(N)$.

Construction (2007)

The map Π_N was independently constructed by Busuioc and S.
Its well-definedness follows via explicit presentation of relative homology and relations of the form $\{x, 1-x\} = 0$ on Steinberg symbols.

Conjecture (S.)

1. *The map Π_N is Eisenstein, i.e., $\Pi_N \circ (T_\ell - 1 - \ell\langle\ell\rangle) = 0$ for all primes ℓ.*
2. *The resulting map ϖ_N on the quotient by I is an isomorphism.*

Work of Fukaya and Kato (2011)

- Proved the first (original) conjecture after tensoring with $\mathbb{Z}_p$ for $p \mid N$. Their method can be extended to $p \nmid N$ if $p \nmid \varphi(N)$.
- Proved a result towards the second conjecture (on p-parts, same conditions) and a stronger p-adic form.

Theorem (S.-Venkatesh)

We have $\Pi_N \circ (T_\ell - 1 - \ell\langle\ell\rangle) = 0$ for all primes $\ell \nmid N$.

Approach of Fukaya-Kato

Method of Fukaya-Kato

Very roughly, for $Y_1(N)$ viewed as a $\mathbb{Z}[\frac{1}{N}]$-scheme, show that Π_N factors as:

$$\begin{array}{ccc} H_1(X_1(N), C_1^\circ(N), \mathbb{Z}) & \xrightarrow{z_N} & K_2(Y_1(N)) \\ & \searrow^{\Pi_N^\circ} & \downarrow \infty \\ & & K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])^+ \end{array} \qquad \begin{array}{ccc} [c:d] & \longmapsto & \{g_{\frac{c}{N}}, g_{\frac{d}{N}}\} \\ & \searrow & \downarrow \\ & & \{1-\zeta_N^c, 1-\zeta_N^d\}. \end{array}$$

Here:

- $\{g_{\frac{c}{N}}, g_{\frac{d}{N}}\}$ are Beilinson-Kato elements, which are Steinberg symbols of Siegel units on $Y_1(N)$,
- z_N is well-defined and Hecke-equivariant by a regulator computation, taking place first up modular and cyclotomic towers,
- ∞ is Eisenstein (for $\ell \mid N$, only on Beilinson-Kato elements).

Remark

The map z_N actually takes values in ordinary cohomology $H^2_{\text{ét}}(Y_1(N), \mathbb{Q}_p(2))^{\text{ord}}$. There is a map $K_2(Y_1(N)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{\text{ét}}(Y_1(N), \mathbb{Z}_p(2))^{\text{ord}}$ with unknown kernel.

Our method

- For the $\mathbb{Q}$-scheme $\mathbb{G}_m^2$, there is a $\mathrm{GL}_2(\mathbb{Z})$-equivariant exact sequence

$$0 \to H^2(\mathbb{G}_m^2, 2) \to K_2(\mathbb{Q}(\mathbb{G}_m^2)) \xrightarrow{\partial} \bigoplus_D \mathbb{Q}(D)^\times \xrightarrow{\partial} \bigoplus_x \mathbb{Z} \to 0$$

  where D runs over divisors and x over closed points, and $H^2(\mathbb{G}_m^2, 2)$ is motivic cohomology. The residue maps ∂ are tame symbols and take orders of zeros in the two cases.

- Associate to $1 \in \mathbb{Z}$ at $x = (1,1)$ a 1-cocycle

$$\Theta \colon \mathrm{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathbb{G}_m^2))/H^2(\mathbb{G}_m^2, 2).$$

- Using the exact sequence, one sees that has an explicit description, is parabolic, integral, and Eisenstein.
- Specialize via pullback by $(1, \zeta_N)$ to obtain a parabolic cocycle

$$\Theta_N : \Gamma_0(N) \to K_2(\mathbb{Z}[\mu_N, \tfrac{1}{N}])/\langle\{-1, -\zeta_N\}\rangle$$

  that is Eisenstein for primes $\ell \nmid N$.
- The restriction of Θ_N to $\Gamma_1(N)$ induces Π_N.

Modular cocycle

- For primes $n \nmid N$, we construct a motivic cocycle
$$ {}_n\Theta\colon\ \mathrm{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathcal{E}^2)) \otimes_{\mathbb{Z}} \mathbb{Z}' $$
for $\mathbb{Z}' = \mathbb{Z}[\frac{1}{5!}]$ for the universal elliptic curve $\mathcal{E}$ over $Y_1(N)$.
- The cocycle ${}_n\Theta$ is parabolic, integral, Hecke-equivariant away from the level, and has an explicit formula in terms of products of theta functions.
- The cocycle ${}_n\Theta$ specializes to a cocycle
$$ {}_n\Theta_N : \Gamma_0(N) \to H^2(Y_1(N), \mathbb{Z}'(2)). $$
- There exists a universal cocycle $\Theta_N : \Gamma_0(N) \to H^2(Y_1(N), \mathbb{Q}(2))$ that gives rise to all ${}_n\Theta_N$.
- Taking $\mathbb{Z}_p$-coefficients and ordinary parts, we recover the maps z_N for $p > 5$ and show their Hecke-equivariance for T_ℓ with $\ell \nmid N$.

Remark

We do not use this construction in studying Π_N.

Motivic cohomology (naive version)

Notation

- Y an equidimensional quasi-projective scheme of finite type over a field F
- Δ^j the j-simplex over F

Definition (Bloch's cycle complex)

Bloch's cycle complex $z^k(Y, \cdot\,)$ has terms

$$z^k(Y,j) = \{\text{pure codim. } k \text{ cycles in } Y \times \Delta^j \text{ meeting faces of } \Delta^j \text{ properly}\}$$

with boundaries given by alternating sums of face maps.

Definition

Set $H^i(Y,k) = H_{2k-i}(z^k(Y, \cdot\,))$ for $i \in \mathbb{Z}$ and $k \geq 0$.

Remark

For Y smooth and F perfect, these are isomorphic to the motivic cohomology groups of Voevodsky.

Properties of motivic cohomology

Properties

- There are pullback and proper pushforward maps.
- If $Y = \coprod_{h=1}^{t} Y_h$, then $H^i(Y,k) = \bigoplus_{h=1}^{t} H^i(Y_h,k)$.
- $H^i(Y,k) \cong H^i(Y \times \mathbb{A}^1, k)$ via pullback.
- $H^0(Y,0) \cong \mathbb{Z}$ if Y is connected, $H^i(Y,0) = 0$ for all $i \neq 0$.
- For Y smooth, $H^1(Y,1) \cong \mathcal{O}_Y^\times$, and $H^i(Y,1) = 0$ for all $i \notin \{1,2\}$.
- For Y smooth, $H^i(Y,k) = 0$ for all $i > k + \dim Y$. If moreover Y separated, then $H^i(Y,k) = 0$ for $i > 2k$.
- For $\rho\colon Z \to Y$ a closed embedding of pure codimension c with open complement $\iota\colon U \to Y$, there is an exact Gysin sequence ($\partial =$ residue map):

$$\cdots \to H^i(Y,k) \xrightarrow{\iota^*} H^i(U,k) \xrightarrow{\partial} H^{i-2c+1}(Z,k-c) \xrightarrow{\rho_*} H^{i+1}(Y,k) \to \cdots .$$

- Products of cycles give rise to external products, and pulling back external products for $Y \times Y$ by the diagonal yields cup products

$$H^i(Y,k) \times H^{i'}(Y,k') \xrightarrow{\cup} H^{i+i'}(Y,k+k').$$

- $\bigoplus_{i=0}^{\infty} H^i(\operatorname{Spec} F, i) \cong \bigoplus_{i=0}^{\infty} K_i^M(F)$, the Milnor K-theory ring. Note that $K_i^M(F) \cong K_i(F)$ for $i \leq 2$.

A coniveau spectral sequence

Coniveau spectral sequence

For $n \geq 0$, there is a right half-plane spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in Y_p} H^{q-p}(k(x), n-p) \Rightarrow H^{p+q}(Y, n),$$

where Y_i denotes the irreducible codimension i cycles on Y (a smooth variety). For $n = 2$, its row for $q = 2$ is a complex K in homological degrees $[2, 0]$:

$$K_2(\mathbb{Q}(Y)) \to \bigoplus_{D \in Y_1} K_1(\mathbb{Q}(D)) \to \bigoplus_{x \in Y_2} K_0(\mathbb{Q}(x)),$$

and we have

$$H_i(\mathsf{K}) \cong H^{4-i}(Y, 2).$$

The case of $\mathbb{G}_m^2$

We have $H^i(\mathbb{G}_m^2, 2) = 0$ for $i > 2$, so there is an exact sequence

$$0 \to H^2(\mathbb{G}_m^2, 2) \to \mathsf{K}_2 \to \mathsf{K}_1 \to \mathsf{K}_0 \to 0.$$

It is equipped with a pullback action of $\Delta = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ induced by the right action of Δ on $\mathbb{G}_m^2$, given on coordinates by $(z_1, z_2)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (z_1^a z_2^c, z_1^b z_2^d)$.

Symbols

Let z_1 and z_2 denote the coordinate functions on $\mathbb{G}_m^2$.

- In K_0, let e be the canonical generator of $H^0(\{1\}, 0) \cong \mathbb{Z}$.
- In K_1, for $a, c \in \mathbb{Z}$ with $(a, c) = 1$, let

  $$\langle a, c \rangle = 1 - z_1^b z_2^d \in H^1(S_{a,c} - \{1\}, 1),$$

  where $ad - bc = 1$ and $S_{a,c} = \ker(\mathbb{G}_m^2 \xrightarrow{(x,y)\mapsto ax+cy} \mathbb{G}_m)$. Then $\langle a, c \rangle = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* \langle 1, 0 \rangle$.
- In K_2, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, let

  $$\langle \gamma \rangle = \langle (a, c), (b, d) \rangle = (1 - z_1^a z_2^c) \cup (1 - z_1^b z_2^d) \in H^2(\mathbb{G}_m^2 - S_{a,c} \cup S_{b,d}, 2).$$

  Then $\langle \gamma \rangle = \gamma^* \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle$.

Residues

We have

$$\partial \langle a, c \rangle = e \quad \text{and} \quad \partial \langle \gamma \rangle = \begin{cases} \langle a, c \rangle - \langle -b, -d \rangle, & \det \gamma = 1 \\ \langle -a, -c \rangle - \langle b, d \rangle, & \det \gamma = -1. \end{cases}$$

The 1-cocycle Θ

Proposition

Set $\overline{\mathrm{K}}_2 = \mathrm{K}_2 / H^2(\mathbb{G}_m^2, 2)$. *There exists a unique* 1*-cocycle*

$$\Theta \colon \mathrm{GL}_2(\mathbb{Z}) \to \overline{\mathrm{K}}_2, \quad \gamma \mapsto \Theta_\gamma$$

such that

$$\partial \Theta_\gamma = (\gamma^* - 1)\langle 0, 1 \rangle.$$

for all $\gamma \in \mathrm{GL}_2(\mathbb{Z})$.

Proof.

For $\gamma, \mu \in \mathrm{GL}_2(\mathbb{Z})$, we have

$$\begin{aligned} \partial \Theta_{\gamma\mu} &= ((\gamma\mu)^* - 1)\langle 0, 1 \rangle \\ &= (\gamma^* - 1)\langle 0, 1 \rangle + \gamma^*(\mu^* - 1)\langle 0, 1 \rangle \\ &= \partial \Theta_\gamma + \gamma^* \partial \Theta_\mu. \end{aligned}$$

Since $\partial \colon \overline{\mathrm{K}}_2 \to \mathrm{K}_1$ is injective and K is $\mathrm{GL}_2(\mathbb{Z})$-equivariant, we have

$$\Theta_{\gamma\mu} = \Theta_\gamma + \gamma^* \Theta_\mu.$$

□

Θ is parabolic

Proposition

The cocycle Θ is parabolic, i.e., $\Theta|_P$ is null-cohomologous on all stabilizers of $\mathbb{P}^1(\mathbb{Q})$ under its right action of $\mathrm{GL}_2(\mathbb{Z})$.

Proof.

Let
$$P = \{\left(\begin{smallmatrix} 1 & 0 \\ c & d \end{smallmatrix}\right) \mid c \in \mathbb{Z},\ d = \pm 1\}.$$
For $\gamma \in P$, we have $\gamma^*\langle 0,1\rangle = \langle 0,1\rangle$, so $\partial\Theta_\gamma = 0$, so $\Theta_\gamma = 0$. Thus $\Theta|_P = 0$. Since the parabolic subgroups form a single conjugacy class, Θ is a coboundary on all of them. □

Explicit formula

Definition

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, a *connecting sequence* $v = (v_i)_{i=0}^k$ for γ is $v_i = (b_i, d_i) \in \mathbb{Z}^2$ such that $v_0 = (0, 1)$, $v_k = \det(\gamma)(b, d)$, and

$$\det \begin{pmatrix} b_{i-1} & b_i \\ d_{i-1} & d_i \end{pmatrix} = 1$$

for all $1 \leq i \leq k$.

Proposition

Let $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ and $v = (v_i)_{i=0}^k$ be a connecting sequence for γ. Then

$$\Theta_\gamma = \sum_{i=1}^{k} \langle v_i, -v_{i-1} \rangle \in \overline{\mathsf{K}}_2.$$

Proof.

$$\partial \left(\sum_{i=1}^{k} \langle v_i, -v_{i-1} \rangle \right) = \sum_{i=1}^{k} (\langle v_i \rangle - \langle v_{i-1} \rangle) = (\gamma^* - 1)\langle 0, 1 \rangle = \partial \Theta_\gamma.$$

□

Notation

Set $\Gamma = \mathrm{GL}_2(\mathbb{Z})$. Fix a prime ℓ. Let $g_0 = \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) \in \Delta = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$, and write

$$\Gamma g_0 \Gamma = \coprod_{j=0}^{\ell} g_j \Gamma$$

with $g_j = \left(\begin{smallmatrix} \ell & j \\ & 1 \end{smallmatrix}\right)$ for $0 \leq j \leq \ell - 1$ and $g_\ell = \left(\begin{smallmatrix} 1 & \\ & \ell \end{smallmatrix}\right)$.
For $\gamma \in \Gamma$, there exists a permutation σ of $\{0, \ldots, \ell\}$ and $\gamma_j \in \Gamma$ with

$$\gamma g_j = g_{\sigma(j)} \gamma_j$$

for $0 \leq j \leq \ell$.

Definition

Let A be a $\mathbb{Z}[\Delta]$-module. If $\theta \colon \Gamma \to A$ is a 1-cocycle, then set

$$T_\ell \theta(\gamma) = \sum_{j=0}^{\ell} g^*_{\sigma(j)} \theta(\gamma_j).$$

This descends to a well-defined action on $H^1(\Gamma, A)$.

Proposition

In $H^1(\mathrm{GL}_2(\mathbb{Z}), \overline{\mathsf{K}}_2)$, the classes of $T_\ell\Theta$ and $(\ell + [\ell]^*)\Theta$ agree.

Proof.

Define T_ℓ on K by $T_\ell = \sum_{j=0}^{\ell} g_j^*$. Then $T_\ell e$ is the sum of the classes of the cyclic subgroups of order ℓ in μ_ℓ^2, and μ_ℓ^2 has class $[\ell]^* e \in \mathsf{K}_0$. That is,

$$T_\ell e = (\ell + [\ell]^*)e.$$

So there exists a unique $\psi \in \overline{\mathsf{K}}_2$ with

$$\partial\psi = (T_\ell - \ell - [\ell]^*)\langle 0, 1\rangle.$$

Since $\gamma^* g_j^* = g_{\sigma(j)}^* \gamma_j^*$, we have

$$\partial(T_\ell\Theta)_\gamma = (\gamma^* - 1)T_\ell\langle 0, 1\rangle,$$

so we have

$$(T_\ell - \ell - [\ell]^*)\Theta_\gamma = (\gamma^* - 1)\psi,$$

which is to say that $(T_\ell - \ell - [\ell]^*)\Theta_\gamma$ is null-cohomologous. □

Definition

For Γ acting on the right on Y, let $H^*_\Gamma(Y,k)$ denote the cohomology of the total complex of the double complex that is the Γ-bar resolution of Bloch's cycle complex $z^k(Y, 2k-\cdot)$. This provides a spectral sequence

$$E_2^{i,j} = H^i(\Gamma, H^j(Y,k)) \Rightarrow H^{i+j}_\Gamma(Y,k).$$

Remark

We set $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ and implicitly tensor everything by $\mathbb{Z}[\frac{1}{6}]$ in what follows. A Gysin sequence gives an isomorphism

$$H^3_\Gamma(\mathbb{G}_m^2 - \{1\}, 2) \xrightarrow{\sim} H^0_\Gamma(\{1\}, 0) \cong \mathbb{Z}.$$

Let $\mathbb{E} \in H^3_\Gamma(\mathbb{G}_m^2 - \{1\}, 2)$ map to the identity class under this isomorphism. The image of $\mathbb{E}$ under the composition

$$H^3_\Gamma(\mathbb{G}_m^2 - \{1\}, 2) \to H^3_\Gamma(\mathbb{Q}(\mathbb{G}_m^2), 2) \to H^1(\Gamma, H^2(\mathbb{Q}(\mathbb{G}_m^2), 2))$$

gives the class of Θ.

Definition

For $m \geq 1$, the *trace map* $[m]_*\colon \mathsf{K} \to \mathsf{K}$ is in degree i the sum of maps

$$[m]_*\colon K_i(\mathbb{Q}(x)) \to \bigoplus_{\substack{y \in Y_{2-i} \\ my = x}} K_i(\mathbb{Q}(y)).$$

for $x \in Y_{2-i}$ given by the norms for the field extensions $\mathbb{Q}(y)/\mathbb{Q}(x)$. Set

$$\mathsf{K}_i^{(0)} = \{c \in \mathsf{K}_i \mid [m]_*(c) = c \text{ for all } m \geq 1\}.$$

Example

Consider $1 - z \in \mathbb{Q}(\mathbb{G}_m)^\times$, where z is the coordinate function on $\mathbb{G}_m$. For any $m \geq 1$, we have

$$[m]_*(1-z) = \prod_{i=0}^{m}(1 - \zeta_m^i z^{1/m}) = 1 - z.$$

Lemma

The symbols e, $\langle a, c\rangle$, *and* $\langle\gamma\rangle$ *lie in* $\mathsf{K}^{(0)}$, *and*

$$H^2(\mathbb{G}_m^2, 2)^{(0)} = \langle -z_1 \cup -z_2\rangle \cong \mathbb{Z}.$$

We therefore have $\Theta\colon \mathrm{GL}_2(\mathbb{Z}) \to \mathsf{K}_2^{(0)}/\langle\{-z_1, -z_2\}\rangle$.

Specialization on motivic cohomology

Let $s\colon \operatorname{Spec}\mathbb{Q}(\mu_N) \to \mathbb{G}_m^2$ with value $(1, \zeta_N) \in \mathbb{G}_m^2(\mathbb{Q}(\mu_N))$, corresponding to

$$\mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}] \to \mathbb{Q}(\mu_N), \quad z_1 \mapsto 1,\ z_2 \mapsto \zeta_N.$$

Let

$$s^*\colon H^2(\mathbb{G}_m^2, 2) \to K_2(\mathbb{Q}(\mu_N)).$$

Then

$$s^*(-z_1 \cup -z_2) = \{-1, -\zeta_N\} = \begin{cases} \{-1, -1\} & N \text{ odd} \\ 0 & N \text{ even.} \end{cases}$$

Remark

The pullback by s doesn't make sense on K_2! There is no $\mathbb{Q}(\mathbb{G}_m^2) \to \mathbb{Q}(\mu_N)$. However, it does make sense on $\varinjlim_{(1,\zeta_N)\in U} H^2(U,2)$ inside K_2.

Congruence subgroup

Let $\Gamma_0 = \tilde{\Gamma}_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \mid N \mid c\}$.

Specialization of Θ_γ

For $\gamma \in \Gamma$, we have $\Theta_\gamma \in H^2(\mathbb{G}_m^2 - S_{0,1} \cup S_{b,d})/\langle\{-z_1, -z_2\}\rangle$. If $\gamma \in \Gamma_0$, then $N \nmid d$, so we may set

$$\Theta_{N,\gamma} = s^*\Theta_\gamma \in K_2(\mathbb{Q}(\mu_N))/\langle\{-1, -\zeta_N\}\rangle.$$

Notation and conventions

- For $N \nmid d$, we let $\sigma_d \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ be such that $\sigma_d(\zeta_N) = \zeta_N^d$.
- We have $\Gamma_0 \to \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \sigma_d$.
- We let Γ_0 act on $K_2(\mathbb{Q}(\mu_N))$ through this map.

Theorem

The map

$$\Theta_N : \tilde{\Gamma}_0(N) \to K_2(\mathbb{Q}(\mu_N))/\langle\{-1, \zeta_N\}\rangle, \quad \gamma \mapsto \Theta_{N,\gamma}$$

is a parabolic 1*-cocycle such that the following hold:*

1. *There exists a connecting sequence* $(b_i, d_i)_{i=0}^k$ *for* γ *with* $N \nmid d_i$ *for all* i*, and*
$$\Theta_{N,\gamma} = \sum_{i=0}^{k} \{1 - \zeta_N^{d_i}, 1 - \zeta_N^{-d_{i-1}}\},$$
2. $(T_\ell - \ell - \sigma_\ell)\Theta_N$ *is null-cohomologous for all primes* $\ell \nmid 2N$*, and if* $2 \nmid N$*, then* $2(T_2 - 2 - \sigma_2)\Theta_N$ *is null-cohomologous.*
3. Θ_N *takes values in* $K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])/\langle\{-1, -\zeta_N\}\rangle$.

Remark

All but the last property follow from the analogous property of Θ. The last is seen from the explicit formula.

Maps on homology

Notation

The restriction of Θ_N to $\Gamma_1 = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0 \mid d \equiv 1 \bmod N \}$ is a homomorphism. Being parabolic, its further restriction to $\Gamma_1(N) = \Gamma_1 \cap \mathrm{SL}_2(\mathbb{Z})$ induces

$$H_1(X_1(N), \mathbb{Z})_+ \to K_2(\mathbb{Z}[\mu_N])/\langle \{-1, \zeta_N\} \rangle,$$

where the subscript $+$ is the maximal quotient on which complex conjugation acts trivially. If we invert 2, we obtain a homomorphism

$$\Pi_N \colon H_1(X_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N])^+.$$

By the explicit formula, it is the restriction of Π_N°, so the map Π_N defined earlier.

Remarks

1. Π_N is $(\mathbb{Z}/N\mathbb{Z})^\times$-equivariant in the sense that for the diamond operator $\langle d \rangle$, we have $\Pi_N \circ \langle d \rangle = \sigma_d \circ \Pi_N$.
2. The theorem tells us that $\Pi_N \circ (T_\ell - \ell - \langle \ell \rangle) = 0$ for $\ell \nmid N$. This appears to differ from our original condition for being Eisenstein, but it is equivalent as we are now using usual rather than dual correspondences (and $\ell \nmid N$).

Square of an elliptic curve

Set-up

Let E be an elliptic curve over Y over a characteristic 0 field F.
Again we have a Δ-equivariant complex K in homological degrees $[2,0]$,

$$K_2(\mathbb{Q}(E^2)) \xrightarrow{\partial} \bigoplus_{D\in(E^2)_1} \mathbb{Q}(D)^\times \xrightarrow{\partial} \bigoplus_{x\in(E^2)_2} \mathbb{Z},$$

with $H_i(\mathsf{K}) = H^{4-i}(E^2, 2)$. None of these groups vanish.

Trace-fixed parts

Fix $n > 1$, and let $\mathbb{N}_n = \{m \geq 1 \mid (m,n) = 1\}$. Let $\mathbb{Z}'$ be a localization of $\mathbb{Z}$.
For a $\mathbb{Z}[\mathbb{N}_n]$-module M, let

$$M^{(0)} = \{x \in M \otimes_{\mathbb{Z}} \mathbb{Z}' \mid [m]_* x = x \text{ for all } m \in \mathbb{N}_n\}.$$

Trace-fixed parts of the cohomology of E^2

For $\mathbb{Z}' = \mathbb{Z}[\frac{1}{6}]$, we have $H^i(E^2, 2)^{(0)} = 0$ unless $i = 4$, in which case it isomorphic to $\mathbb{Z}'$. This arises from a slight extension of work of Deninger-Murre to allow integral coefficients, and involves the Fourier-Mukai transform on E^2.

Remark

The sequence $0 \to \mathsf{K}_2^{(0)} \to \mathsf{K}_1^{(0)} \to \mathsf{K}_0^{(0)} \to \mathbb{Z}' \to 0$ is exact outside of $\mathsf{K}_0^{(0)}$.

Construction of an abstract cocycle

Let $Z \in \ker(\mathsf{K}_0^{(0)} \to \mathbb{Z}')$ be $\mathrm{GL}_2(\mathbb{Z})$-fixed. If it is the image of some $\eta \in \mathsf{K}_1^{(0)}$, then we can define

$$\Theta^Z\colon\ \mathrm{GL}_2(\mathbb{Z}) \to \mathsf{K}_2^{(0)}, \qquad \gamma \mapsto \Theta_\gamma^Z$$

for $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ by

$$\partial\Theta_\gamma^Z = (\gamma^* - 1)\eta.$$

Cycles for the universal elliptic curve

For $\mathcal{E}$ the universal elliptic curve over $Y = Y_1(N)$ over $\mathbb{Q}$ with $N \geq 4$,

$$e_n = n(n^3(0) - nT_n(0) + \mathcal{E}[n]^2) \in \mathsf{K}_0^{(0)}$$

is $\mathrm{GL}_2(\mathbb{Z})$-fixed and the residue of an element $\langle 0, 1\rangle_n \in \mathsf{K}_1^{(0)}$ formed out of theta-functions on $\mathcal{E}$ and their divisors. Hence, we obtain a cocycle ${}_n\Theta$.

Remarks

The cocycle

- is parabolic,
- satisfies an explicit formula for sums of symbols formed out of exterior products of theta-functions,
- is equivariant in the sense that the class of $T_\ell({}_n\Theta)$ for $\ell \nmid N$ equals the class of $T'_\ell({}_n\Theta)$ for T'_ℓ determined by a correspondence on Y. (Here, we also need to invert 5 for $\ell = 5$, so take $\mathbb{Z}' = \mathbb{Z}[\frac{1}{30}]$ from now on.)

There is no universal cocycle independent of n, much as with theta-functions. However, setting

$$V_\ell = \ell(\ell^3 - \ell T_\ell + [\ell]^*),$$

the classes $[V_\ell({}_n\Theta)]$ and $[V_n({}_\ell\Theta)]$ are equal.

Specialized cocycles

Specialization

On $\tilde{\Gamma}_0(N)$, we can pull back ${}_n\Theta$ by $s = (0, \iota)$ with $\iota\colon Y \to \mathcal{E}$ the canonical N-torsion section to obtain a Hecke-equivariant parabolic cocycle

$$ {}_n\Theta_N\colon \tilde{\Gamma}_0(N) \to H^2(Y, \mathbb{Z}'(2)). $$

Universal cocycle

Much as with Siegel units, there exists $\Theta_N\colon \tilde{\Gamma}_0(N) \to H^2(Y, \mathbb{Z}'[\frac{1}{N}](2))$ satisfying $[V_n(\Theta_N)] = [{}_n\Theta_N]$. For $\gamma \in \tilde{\Gamma}_1(N)$ and an N-connecting sequence $(b_i, d_i)_{i=0}^k$, we have

$$ \Theta_{N,\gamma} \equiv \sum_{i=1}^{k} g_{\frac{d_i}{N}} \cup g_{\frac{-d_{i-1}}{N}} \bmod \mathcal{V}, $$

where $g_{\frac{u}{N}}$ is the usual Siegel unit on Y for $N \nmid u$, and $\mathcal{V}$ is the common kernel of all (analogously-defined) operators V'_ℓ on $H^2(Y, \mathbb{Z}'(2))$.

Remarks

- On $\tilde{\Gamma}_1(N)$, these cocycles actually take values in the cohomology of $X_1(N)$.
- The group $\mathcal{V}$ vanishes in any standard realization.

Zeta maps

The motivic zeta map

The map Θ_N induces a zeta map

$$z_N : H_1(X_1(N), \mathbb{Z})_+ \to H^2(Y, \mathbb{Z}'[\tfrac{1}{N}](2))$$

satisfying $z_N \circ T_\ell = T_\ell \circ z_N$ for $\ell \nmid N$.

Comparison with known constructions

- The composition of z_N (defined over $\mathbb{Z}[\frac{1}{6N}]$) with the map to $K_2(Y) \otimes \mathbb{Z}[\frac{1}{6N}]$ agrees with maps of Goncharov and Brunault (modulo the image of $\mathcal{V}$).
- The composition of z_N with the map to $H^2_{\text{ét}}(Y, \mathbb{Q}_p(2))^{\text{ord}}$ agrees with a map of Fukaya-Kato for $p \mid N$ up to an Atkin-Lehner involution. They show their map to be equivariant for all Hecke-operators (using dual operators on the right) via a regulator computation. (For $p \nmid N$, it agrees with a map of Lecouturier and J. Wang.)

p-adic integrality

We can actually construct a zeta map z_N to $H^2_{\text{ét}}(Y, \mathbb{Z}_p(2))^{\text{ord}}$ after removing the $(\mathbb{Z}/p\mathbb{Z})^\times$-eigenspace for the square of $\omega\colon (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times$.