Eisenstein cocycles in motivic cohomology

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The maps

Goals

1. To give a new construction of explicit maps of Busuioc and S. for $N \geq 1$:

$$\Pi_N : H_1(X_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N])^+, \quad [c : d] \mapsto \{1 - \zeta_N^c, 1 - \zeta_N^d\}.$$

taking (projections of) Manin symbols to Steinberg symbols.

Here, $+$ denotes the part fixed by complex conjugation after inverting 2.

2. To verify that $\Pi_N$ is Eiseinstein, i.e., factors through the quotient of homology by an Eisenstein ideal $I$ in the weight 2 Hecke algebra.

Details

- The symbols in question lie in a homology group relative to cusps $C_1^0(N)$ not over $\infty \in X_0(N)$ and the second $K$-group of $\mathbb{Z}[\mu_N, \frac{1}{N}]$. We define a map $\Pi_N^\circ$ on these groups and restrict.

- The Manin symbols are classes of geodesics $[c : d] = \{\frac{a}{c} \to \frac{b}{d}\}$ between cusps, where $ad - bc = 1$. They depend only on (nonzero) $c, d$ modulo $N$.

- The Steinberg symbols $\{1 - \zeta_N^c, 1 - \zeta_N^d\}$ are of cyclotomic $N$-units. Here, $\zeta_N = e^{2\pi i/N}$, viewing $\overline{\mathbb{Q}} \subset \mathbb{C}$.

- The Eisenstein ideal $I$ is generated by $T_\ell - 1 - \ell \langle \ell \rangle$ for primes $\ell$, where we take $\langle \ell \rangle = 0$ if $\ell \nmid N$. The action is via dual correspondences on $X_1(N)$. 


Background and Result

Construction (2007)

The map $\Pi_N$ was independently constructed by Busuioc and S. Its well-definedness follows via explicit presentation of relative homology and relations of the form $\{x, 1 - x\} = 0$ on Steinberg symbols.

Conjecture (S.)

1. The map $\Pi_N$ is Eisenstein, i.e., $\Pi_N \circ (T_\ell - 1 - \ell \langle \ell \rangle) = 0$ for all primes $\ell$.
2. The resulting map $\varpi_N$ on the quotient by $I$ is an isomorphism.

Work of Fukaya and Kato (2011)

- Proved the first (original) conjecture after tensoring with $\mathbb{Z}_p$ for $p \mid N$. Their method can be extended to $p \nmid N$ if $p \nmid \varphi(N)$.
- Proved a result towards the second conjecture (on $p$-parts, same conditions) and a stronger $p$-adic form.

Theorem (S.-Venkatesh)

We have $\Pi_N \circ (T_\ell - 1 - \ell \langle \ell \rangle) = 0$ for all primes $\ell \nmid N$. 


Approach of Fukaya-Kato

Method of Fukaya-Kato

Very roughly, for $Y_1(N)$ viewed as a $\mathbb{Z}[\frac{1}{N}]$-scheme, show that $\Pi_N$ factors as:

$$H_1(X_1(N), C_1^0(N), \mathbb{Z}) \xrightarrow{z_N} K_2(Y_1(N)) \xrightarrow{\Pi^0_N} K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])^+ \xrightarrow{\infty} \{1 - \zeta^c_N, 1 - \zeta^d_N\}.$$  

Here:

- $\{g_{\frac{c}{N}}, g_{\frac{d}{N}}\}$ are Beilinson-Kato elements, which are Steinberg symbols of Siegel units on $Y_1(N)$,
- $z_N$ is well-defined and Hecke-equivariant by a regulator computation, taking place first up modular and cyclotomic towers,
- $\infty$ is Eisenstein (for $\ell \mid N$, only on Beilinson-Kato elements).

Remark

The map $z_N$ actually takes values in ordinary cohomology $H^2_{\text{ét}}(Y_1(N), \mathbb{Q}_p(2))^{\text{ord}}$. There is a map $K_2(Y_1(N)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{\text{ét}}(Y_1(N), \mathbb{Z}_p(2))^{\text{ord}}$ with unknown kernel.
For the $\mathbb{Q}$-scheme $\mathbb{G}_m^2$, there is a $\text{GL}_2(\mathbb{Z})$-equivariant exact sequence

$$0 \to H^2(\mathbb{G}_m^2, 2) \to K_2(\mathbb{Q}(\mathbb{G}_m^2)) \xrightarrow{\partial} \bigoplus_D \mathbb{Q}(D)^\times \xrightarrow{\partial} \bigoplus_x \mathbb{Z} \to 0$$

where $D$ runs over divisors and $\times$ over closed points, and $H^2(\mathbb{G}_m^2, 2)$ is motivic cohomology. The residue maps $\partial$ are tame symbols and take orders of zeros in the two cases.

- Associate to $1 \in \mathbb{Z}$ at $x = (1, 1)$ a 1-cocycle

$$\Theta : \text{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathbb{G}_m^2))/H^2(\mathbb{G}_m^2, 2).$$

- Using the exact sequence, one sees that has an explicit description, is parabolic, integral, and Eisenstein.

- Specialize via pullback by $(1, \zeta_N)$ to obtain a parabolic cocycle

$$\Theta_N : \Gamma_0(N) \to K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])/\langle\{1, -\zeta_N\}\rangle$$

that is Eisenstein for primes $\ell \nmid N$.

- The restriction of $\Theta_N$ to $\Gamma_1(N)$ induces $\Pi_N$. 
For primes $n \nmid N$, we construct a motivic cocycle

$$n \Theta : \text{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(E^2)) \otimes_{\mathbb{Z}} \mathbb{Z}'$$

for $\mathbb{Z}' = \mathbb{Z}[\frac{1}{5!}]$ for the universal elliptic curve $E$ over $Y_1(N)$.

The cocycle $n \Theta$ is parabolic, integral, Hecke-equivariant away from the level, and has an explicit formula in terms of products of theta functions.

The cocycle $n \Theta$ specializes to a cocycle

$$n \Theta_N : \Gamma_0(N) \to H^2(Y_1(N), \mathbb{Z}'(2)).$$

There exists a universal cocycle $\Theta_N : \Gamma_0(N) \to H^2(Y_1(N), \mathbb{Q}(2))$ that gives rise to all $n \Theta_N$.

Taking $\mathbb{Z}_p$-coefficients and ordinary parts, we recover the maps $z_N$ for $p > 5$ and show their Hecke-equivariance for $T_\ell$ with $\ell \nmid N$.

Remark

We do not use this construction in studying $\Pi_N$. 
Motivic cohomology (naive version)

**Notation**
- $Y$ an equidimensional quasi-projective scheme of finite type over a field $F$
- $\Delta^j$ the $j$-simplex over $F$

**Definition (Bloch’s cycle complex)**

Bloch’s cycle complex $z^k(Y, \cdot)$ has terms

$$z^k(Y, j) = \{ \text{pure codim. } k \text{ cycles in } Y \times \Delta^j \text{ meeting faces of } \Delta^j \text{ properly} \}$$

with boundaries given by alternating sums of face maps.

**Definition**

Set $H^i(Y, k) = H_{2k-i}(z^k(Y, \cdot))$ for $i \in \mathbb{Z}$ and $k \geq 0$.

**Remark**

For $Y$ smooth and $F$ perfect, these are isomorphic to the motivic cohomology groups of Voevodsky.
Properties of motivic cohomology

Properties

- There are pullback and proper pushforward maps.
- If \( Y = \bigsqcup_{h=1}^{t} Y_h \), then \( H^i(Y, k) = \bigoplus_{h=1}^{t} H^i(Y_h, k) \).
- \( H^i(Y, k) \cong H^i(Y \times \mathbb{A}^1, k) \) via pullback.
- \( H^0(Y, 0) \cong \mathbb{Z} \) if \( Y \) is connected, \( H^i(Y, 0) = 0 \) for all \( i \neq 0 \).
- For \( Y \) smooth, \( H^1(Y, 1) \cong \mathcal{O}_Y^\times \), and \( H^i(Y, 1) = 0 \) for all \( i \notin \{1, 2\} \).
- For \( Y \) smooth, \( H^i(Y, k) = 0 \) for all \( i > k + \dim Y \). If moreover \( Y \) separated, then \( H^i(Y, k) = 0 \) for \( i > 2k \).
- For \( \rho: Z \to Y \) a closed embedding of pure codimension \( c \) with open complement \( \iota: U \to Y \), there is an exact Gysin sequence (\( \partial = \) residue map):

\[
\cdots \to H^i(Y, k) \xrightarrow{\iota^*} H^i(U, k) \xrightarrow{\partial} H^{i-2c+1}(Z, k-c) \xrightarrow{\rho^*} H^{i+1}(Y, k) \to \cdots.
\]

- Products of cycles give rise to external products, and pulling back external products for \( Y \times Y \) by the diagonal yields cup products

\[
H^i(Y, k) \times H^{i'}(Y, k') \xrightarrow{\cup} H^{i+i'}(Y, k+k').
\]

- \( \bigoplus_{i=0}^{\infty} H^i(\text{Spec } F, i) \cong \bigoplus_{i=0}^{\infty} K^M_i(F) \), the Milnor \( K \)-theory ring.
  Note that \( K^M_i(F) \cong K_i(F) \) for \( i \leq 2 \).
A coniveau spectral sequence

Coniveau spectral sequence

For $n \geq 0$, there is a right half-plane spectral sequence

$$E_{p,q}^1 = \bigoplus_{x \in Y_p} H^{q-p}(k(x), n - p) \Rightarrow H^{p+q}(Y, n),$$

where $Y_i$ denotes the irreducible codimension $i$ cycles on $Y$ (a smooth variety). For $n = 2$, its row for $q = 2$ is a complex $K$ in homological degrees $[2, 0]$:

$$K_2(\mathbb{Q}(Y)) \rightarrow \bigoplus_{D \in Y_1} K_1(\mathbb{Q}(D)) \rightarrow \bigoplus_{x \in Y_2} K_0(\mathbb{Q}(x)),$$

and we have

$$H_i(K) \cong H^{4-i}(Y, 2).$$

The case of $\mathbb{G}_m^2$

We have $H^i(\mathbb{G}_m^2, 2) = 0$ for $i > 2$, so there is an exact sequence

$$0 \rightarrow H^2(\mathbb{G}_m^2, 2) \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow 0.$$ 

It is equipped with a pullback action of $\Delta = M_2(\mathbb{Z}) \cap \text{GL}_2(\mathbb{Q})$ induced by the right action of $\Delta$ on $\mathbb{G}_m^2$, given on coordinates by $(z_1, z_2) ( \begin{pmatrix} a & b \\ c & d \end{pmatrix} ) = (z_1^a z_2^c, z_1^b z_2^d)$. 
Symbols

Let $z_1$ and $z_2$ denote the coordinate functions on $\mathbb{G}_m^2$.

- In $K_0$, let $e$ be the canonical generator of $H^0(\{1\}, 0) \cong \mathbb{Z}$.
- In $K_1$, for $a, c \in \mathbb{Z}$ with $(a, c) = 1$, let

$$\langle a, c \rangle = 1 - z_1^b z_2^d \in H^1(S_{a,c} - \{1\}, 1),$$

where $ad - bc = 1$ and $S_{a,c} = \ker(\mathbb{G}_m^2 \xrightarrow{(x,y) \mapsto ax+cy} \mathbb{G}_m)$. Then $\langle a, c \rangle = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})^* \langle 1, 0 \rangle$.

- In $K_2$, for $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{GL}_2(\mathbb{Z})$, let

$$\langle \gamma \rangle = \langle (a, c), (b, d) \rangle = (1 - z_1^a z_2^c) \cup (1 - z_1^b z_2^d) \in H^2(\mathbb{G}_m^2 - S_{a,c} \cup S_{b,d}, 2).$$

Then $\langle \gamma \rangle = \gamma^* \langle (1 \ 0) \rangle$.

Residues

We have

$$\partial \langle a, c \rangle = e \quad \text{and} \quad \partial \langle \gamma \rangle = \begin{cases} \langle a, c \rangle - \langle -b, -d \rangle, & \det \gamma = 1 \\ \langle -a, -c \rangle - \langle b, d \rangle, & \det \gamma = -1. \end{cases}$$
The 1-cocycle $\Theta$

Proposition

Set $\overline{K}_2 = K_2 / H^2(\mathbb{G}_m^2, 2)$. There exists a unique 1-cocycle

$\Theta : \text{GL}_2(\mathbb{Z}) \to \overline{K}_2, \quad \gamma \mapsto \Theta_\gamma$

such that

$\partial \Theta_\gamma = (\gamma^* - 1)\langle 0, 1 \rangle$. 

for all $\gamma \in \text{GL}_2(\mathbb{Z})$.

Proof.

For $\gamma, \mu \in \text{GL}_2(\mathbb{Z})$, we have

$\partial \Theta_{\gamma\mu} = ((\gamma\mu)^* - 1)\langle 0, 1 \rangle$

$= (\gamma^* - 1)\langle 0, 1 \rangle + \gamma^* (\mu^* - 1)\langle 0, 1 \rangle$

$= \partial \Theta_\gamma + \gamma^* \partial \Theta_\mu$.

Since $\partial : \overline{K}_2 \to K_1$ is injective and $K$ is $\text{GL}_2(\mathbb{Z})$-equivariant, we have

$\Theta_{\gamma\mu} = \Theta_\gamma + \gamma^* \Theta_\mu$. 

Proposition

The cocycle $\Theta$ is parabolic, i.e., $\Theta|_P$ is null-cohomologous on all stabilizers of $\mathbb{P}^1(\mathbb{Q})$ under its right action of $GL_2(\mathbb{Z})$.

Proof.

Let

$$P = \{(\frac{1}{c} \ 0 \ d) \mid c \in \mathbb{Z}, \ d = \pm 1\}.$$

For $\gamma \in P$, we have $\gamma^* \langle 0, 1 \rangle = \langle 0, 1 \rangle$, so $\partial \Theta_\gamma = 0$, so $\Theta_\gamma = 0$. Thus $\Theta|_P = 0$. Since the parabolic subgroups form a single conjugacy class, $\Theta$ is a coboundary on all of them.
Definition

For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) \), a connecting sequence \( v = (v_i)_{i=0}^k \) for \( \gamma \) is \( v_i = (b_i, d_i) \in \mathbb{Z}^2 \) such that \( v_0 = (0, 1) \), \( v_k = \det(\gamma)(b, d) \), and

\[
\det \begin{pmatrix} b_{i-1} & b_i \\ d_{i-1} & d_i \end{pmatrix} = 1
\]

for all \( 1 \leq i \leq k \).

Proposition

Let \( \gamma \in \text{GL}_2(\mathbb{Z}) \) and \( v = (v_i)_{i=0}^k \) be a connecting sequence for \( \gamma \). Then

\[
\Theta_\gamma = \sum_{i=1}^k \langle v_i, -v_{i-1} \rangle \in \overline{K}_2.
\]

Proof.

\[
\partial \left( \sum_{i=1}^k \langle v_i, -v_{i-1} \rangle \right) = \sum_{i=1}^k (\langle v_i \rangle - \langle v_{i-1} \rangle) = (\gamma^* - 1) \langle 0, 1 \rangle = \partial \Theta_\gamma.
\]
Notation

Set $\Gamma = \text{GL}_2(\mathbb{Z})$. Fix a prime $\ell$. Let $g_0 = \left( \frac{\ell}{1} \right) \in \Delta = M_2(\mathbb{Z}) \cap \text{GL}_2(\mathbb{Q})$, and write

$$\Gamma g_0 \Gamma = \bigsqcup_{j=0}^{\ell} g_j \Gamma$$

with $g_j = \left( \frac{\ell}{j} \right)$ for $0 \leq j \leq \ell - 1$ and $g_\ell = \left( \frac{1}{\ell} \right)$.

For $\gamma \in \Gamma$, there exists a permutation $\sigma$ of $\{0, \ldots, \ell\}$ and $\gamma_j \in \Gamma$ with

$$\gamma g_j = g_{\sigma(j)} \gamma_j$$

for $0 \leq j \leq \ell$.

Definition

Let $A$ be a $\mathbb{Z}[\Delta]$-module. If $\theta: \Gamma \to A$ is a 1-cocycle, then set

$$T_\ell \theta(\gamma) = \sum_{j=0}^{\ell} g_{\sigma(j)}^* \theta(\gamma_j).$$

This descends to a well-defined action on $H^1(\Gamma, A)$. 

14 / 26
Proposition

In \( H^1(\text{GL}_2(\mathbb{Z}), \overline{K}_2) \), the classes of \( T_\ell \Theta \) and \( (\ell + [\ell]^*) \Theta \) agree.

Proof.

Define \( T_\ell \) on \( K \) by \( T_\ell = \sum_{j=0}^\ell g_j^* \). Then \( T_\ell e \) is the sum of the classes of the cyclic subgroups of order \( \ell \) in \( \mu^2_\ell \), and \( \mu^2_\ell \) has class \( [\ell]^* e \in K_0 \). That is,

\[
T_\ell e = (\ell + [\ell]^*) e.
\]

So there exists a unique \( \psi \in \overline{K}_2 \) with

\[
\partial \psi = (T_\ell - \ell - [\ell]^*) \langle 0, 1 \rangle.
\]

Since \( \gamma^* g_j^* = g_{\sigma(j)}^* \gamma_j^* \), we have

\[
\partial (T_\ell \Theta) \gamma = (\gamma^* - 1) T_\ell \langle 0, 1 \rangle,
\]

so we have

\[
(T_\ell - \ell - [\ell]^*) \Theta \gamma = (\gamma^* - 1) \psi,
\]

which is to say that \( (T_\ell - \ell - [\ell]^*) \Theta \gamma \) is null-cohomologous.
Equivariant class

Definition

For \( \Gamma \) acting on the right on \( Y \), let \( H^*_{\Gamma}(Y, k) \) denote the cohomology of the total complex of the double complex that is the \( \Gamma \)-bar resolution of Bloch’s cycle complex \( z^k(Y, 2k - \cdot) \). This provides a spectral sequence

\[
E^{i,j}_2 = H^i(\Gamma, H^j(Y, k)) \Rightarrow H^{i+j}_{\Gamma}(Y, k).
\]

Remark

We set \( \Gamma = \text{GL}_2(\mathbb{Z}) \) and implicitly tensor everything by \( \mathbb{Z}[\frac{1}{6}] \) in what follows. A Gysin sequence gives an isomorphism

\[
H^3_{\Gamma}(\mathbb{G}_m^2 - \{1\}, 2) \sim\rightarrow H^0_{\Gamma}(\{1\}, 0) \cong \mathbb{Z}.
\]

Let \( E \in H^3_{\Gamma}(\mathbb{G}_m^2 - \{1\}, 2) \) map to the identity class under this isomorphism. The image of \( E \) under the composition

\[
H^3_{\Gamma}(\mathbb{G}_m^2 - \{1\}, 2) \to H^3_{\Gamma}(\mathbb{Q}(\mathbb{G}_m^2), 2) \to H^1(\Gamma, H^2(\mathbb{Q}(\mathbb{G}_m^2), 2))
\]

gives the class of \( \Theta \).
### Definition

For $m \geq 1$, the trace map $[m]_* : K \to K$ is in degree $i$ the sum of maps

$$[m]_* : K_i(\mathbb{Q}(x)) \to \bigoplus_{\substack{y \in Y_{2-i} \\ my = x}} K_i(\mathbb{Q}(y)).$$

for $x \in Y_{2-i}$ given by the norms for the field extensions $\mathbb{Q}(y)/\mathbb{Q}(x)$. Set

$$K_i^{(0)} = \{c \in K_i \mid [m]_* (c) = c \text{ for all } m \geq 1\}.$$

### Example

Consider $1 - z \in \mathbb{Q}(\mathbb{G}_m)^\times$, where $z$ is the coordinate function on $\mathbb{G}_m$. For any $m \geq 1$, we have

$$[m]_* (1 - z) = \prod_{i=0}^{m} (1 - \zeta_m^i z^{1/m}) = 1 - z.$$
**Lemma**

The symbols $e$, $\langle a, c \rangle$, and $\langle \gamma \rangle$ lie in $K^{(0)}$, and

$$H^2(\mathbb{G}_m^2, 2)^{(0)} = \langle -z_1 \cup -z_2 \rangle \cong \mathbb{Z}.$$ 

We therefore have $\Theta : \text{GL}_2(\mathbb{Z}) \to K_2^{(0)} / \langle \{ -z_1, -z_2 \} \rangle$.

**Specialization on motivic cohomology**

Let $s : \text{Spec } \mathbb{Q}(\mu_N) \to \mathbb{G}_m^2$ with value $(1, \zeta_N) \in \mathbb{G}_m^2(\mathbb{Q}(\mu_N))$, corresponding to

$$\mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}] \to \mathbb{Q}(\mu_N), \quad z_1 \mapsto 1, \ z_2 \mapsto \zeta_N.$$

Let

$$s^* : H^2(\mathbb{G}_m^2, 2) \to K_2(\mathbb{Q}(\mu_N)).$$

Then

$$s^*( -z_1 \cup -z_2 ) = \{ -1, -\zeta_N \} = \begin{cases} 
\{-1, -1\} & N \text{ odd} \\
0 & N \text{ even.}
\end{cases}$$
Remark

The pullback by $s$ doesn’t make sense on $K_2$! There is no $\mathbb{Q}(G_m^2) \to \mathbb{Q}(\mu_N)$. However, it does make sense on $\lim_{\gamma(1,\zeta_N) \in U} H^2(U, 2)$ inside $K_2$.

Congruence subgroup

Let $\Gamma_0 = \tilde{\Gamma}_0(N) = \{(a\, b) \in \text{GL}_2(\mathbb{Z}) \mid N \mid c\}$.

Specialization of $\Theta_\gamma$

For $\gamma \in \Gamma$, we have $\Theta_\gamma \in H^2(G_m^2 - S_{0,1} \cup S_{b,d})/\langle \{-z_1, -z_2\}\rangle$. If $\gamma \in \Gamma_0$, then $N \nmid d$, so we may set

$$\Theta_{N,\gamma} = s^* \Theta_\gamma \in K_2(\mathbb{Q}(\mu_N))/\langle \{-1, -\zeta_N\}\rangle.$$  

Notation and conventions

- For $N \nmid d$, we let $\sigma_d \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ be such that $\sigma_d(\zeta_N) = \zeta_N^d$.
- We have $\Gamma_0 \to \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ given by $(a\, b) \mapsto \sigma_d$.
- We let $\Gamma_0$ act on $K_2(\mathbb{Q}(\mu_N))$ through this map.
The main theorem

Theorem

The map

$$\Theta_N: \tilde{\Gamma}_0(N) \to K_2(\mathbb{Q}(\mu_N))/\langle\{-1, \zeta_N\}\rangle, \quad \gamma \mapsto \Theta_N,\gamma$$

is a parabolic 1-cocycle such that the following hold:

1. There exists a connecting sequence \((b_i, d_i)\) for \(\gamma\) with \(N \nmid d_i\) for all \(i\), and

   \[ \Theta_{N,\gamma} = \sum_{i=0}^{k} \{1 - \zeta_N^{d-i}, 1 - \zeta_N^{d-i-1}\}, \]

2. \((T_\ell - \ell - \sigma_\ell)\Theta_N\) is null-cohomologous for all primes \(\ell \nmid 2N\), and if \(2 \nmid N\), then \(2(T_2 - 2 - \sigma_2)\Theta_N\) is null-cohomologous.

3. \(\Theta_N\) takes values in \(K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])/\langle\{-1, -\zeta_N\}\rangle\).

Remark

All but the last property follow from the analogous property of \(\Theta\). The last is seen from the explicit formula.
Maps on homology

Notation

The restriction of $\Theta_N$ to $\Gamma_1 = \{(a \ b) \in \Gamma_0 \mid d \equiv 1 \mod N\}$ is a homomorphism. Being parabolic, its further restriction to $\Gamma_1(N) = \Gamma_1 \cap \text{SL}_2(\mathbb{Z})$ induces

$$H_1(X_1(N), \mathbb{Z})_+ \to K_2(\mathbb{Z}[\mu_N])/\langle\{-1, \zeta_N\}\rangle,$$

where the subscript $+$ is the maximal quotient on which complex conjugation acts trivially. If we invert 2, we obtain a homomorphism

$$\Pi_N : H_1(X_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N])^+.$$ 

By the explicit formula, it is the restriction of $\Pi_N^\circ$, so the map $\Pi_N$ defined earlier.

Remarks

1. $\Pi_N$ is $(\mathbb{Z}/N\mathbb{Z})^\times$-equivariant in the sense that for the diamond operator $\langle d \rangle$, we have $\Pi_N \circ \langle d \rangle = \sigma_d \circ \Pi_N$.
2. The theorem tells us that $\Pi_N \circ (T_\ell - \ell - \langle \ell \rangle) = 0$ for $\ell \nmid N$. This appears to differ from our original condition for being Eisenstein, but it is equivalent as we are now using usual rather than dual correspondences (and $\ell \nmid N$).
Set-up

Let $E$ be an elliptic curve over $Y$ over a characteristic 0 field $F$. Again we have a $\Delta$-equivariant complex $K$ in homological degrees $[2, 0]$,

$$K_2(\mathbb{Q}(E^2)) \xrightarrow{\partial} \bigoplus_{D \in (E^2)_1} \mathbb{Q}(D) \times \bigoplus_{x \in (E^2)_2} \mathbb{Z},$$

with $H_i(K) = H^{4-i}(E^2, 2)$. None of these groups vanish.

Trace-fixed parts

Fix $n > 1$, and let $\mathbb{N}_n = \{m \geq 1 \mid (m, n) = 1\}$. Let $\mathbb{Z}'$ be a localization of $\mathbb{Z}$. For a $\mathbb{Z}[\mathbb{N}_n]$-module $M$, let

$$M^{(0)} = \{x \in M \otimes_{\mathbb{Z}} \mathbb{Z}' \mid [m]_* x = x \text{ for all } m \in \mathbb{N}_n\}.$$ 

Trace-fixed parts of the cohomology of $E^2$

For $\mathbb{Z}' = \mathbb{Z}[\frac{1}{6}]$, we have $H_i(E^2, 2)^{(0)} = 0$ unless $i = 4$, in which case it isomorphic to $\mathbb{Z}'$. This arises from a slight extension of work of Deninger-Murre to allow integral coefficients, and involves the Fourier-Mukai transform on $E^2$. 

Remark
The sequence $0 \to K_2(0) \to K_1(0) \to K_0(0) \to \mathbb{Z}' \to 0$ is exact outside of $K_0(0)$.

Construction of an abstract cocycle
Let $Z \in \ker(K_0(0) \to \mathbb{Z}')$ be $\text{GL}_2(\mathbb{Z})$-fixed. If it is the image of some $\eta \in K_1(0)$, then we can define
\[
\Theta^Z : \text{GL}_2(\mathbb{Z}) \to K_2(0), \quad \gamma \mapsto \Theta^Z_{\gamma}
\]
for $\gamma \in \text{GL}_2(\mathbb{Z})$ by
\[
\partial \Theta^Z_{\gamma} = (\gamma^* - 1)\eta.
\]

Cocycles for the universal elliptic curve
For $E$ the universal elliptic curve over $Y = Y_1(N)$ over $\mathbb{Q}$ with $N \geq 4$,
\[
e_n = n(n^3(0) - nT_n(0) + E[n]^2) \in K_0(0)
\]
is $\text{GL}_2(\mathbb{Z})$-fixed and the residue of an element $\langle 0, 1 \rangle_n \in K_1(0)$ formed out of theta-functions on $E$ and their divisors. Hence, we obtain a cocycle $\nu \Theta$. 
The cocycle $n\Theta$

Remarks

The cocycle

- is parabolic,
- satisfies an explicit formula for sums of symbols formed out of exterior products of theta-functions,
- is equivariant in the sense that the class of $T_\ell(n\Theta)$ for $\ell \nmid N$ equals the class of $T'_\ell(n\Theta)$ for $T'_\ell$ determined by a correspondence on $Y$. (Here, we also need to invert 5 for $\ell = 5$, so take $\mathbb{Z}' = \mathbb{Z}[\frac{1}{30}]$ from now on.)

There is no universal cocycle independent of $n$, much as with theta-functions. However, setting

$$V_\ell = \ell(\ell^3 - \ell T_\ell + [\ell]^*),$$

the classes $[V_\ell(n\Theta)]$ and $[V_n(\ell\Theta)]$ are equal.
Specialized cocycles

Specialization

On $\tilde{\Gamma}_0(N)$, we can pull back $n\Theta$ by $s = (0, \iota)$ with $\iota: Y \to \mathcal{E}$ the canonical $N$-torsion section to obtain a Hecke-equivariant parabolic cocycle

$$n\Theta_N : \tilde{\Gamma}_0(N) \to H^2(Y, \mathbb{Z}'(2)).$$

Universal cocycle

Much as with Siegel units, there exists $\Theta_N : \tilde{\Gamma}_0(N) \to H^2(Y, \mathbb{Z}'[\frac{1}{N}](2))$ satisfying $[V_n(\Theta_N)] = [n\Theta_N]$. For $\gamma \in \tilde{\Gamma}_1(N)$ and an $N$-connecting sequence $(b_i, d_i)_{i=0}^k$, we have

$$\Theta_N, \gamma \equiv \sum_{i=1}^k g_{\frac{d_i}{N}} \cup g_{-\frac{d_{i-1}}{N}} \mod \mathcal{V},$$

where $g_{\frac{u}{N}}$ is the usual Siegel unit on $Y$ for $N \nmid u$, and $\mathcal{V}$ is the common kernel of all (analogously-defined) operators $V'_\ell$ on $H^2(Y, \mathbb{Z}'(2))$.

Remarks

- On $\tilde{\Gamma}_1(N)$, these cocycles actually take values in the cohomology of $X_1(N)$.
- The group $\mathcal{V}$ vanishes in any standard realization.
The motivic zeta map

The map $\Theta_N$ induces a zeta map

$$z_N : H_1(X_1(N), \mathbb{Z})_+ \to H^2(Y, \mathbb{Z}' \left[ \frac{1}{N} \right](2))$$

satisfying $z_N \circ T_\ell = T_\ell \circ z_N$ for $\ell \nmid N$.

Comparison with known constructions

- The composition of $z_N$ (defined over $\mathbb{Z}[\frac{1}{6N}]$) with the map to $K_2(Y) \otimes \mathbb{Z}[\frac{1}{6N}]$ agrees with maps of Goncharov and Brunault (modulo the image of $\mathcal{V}$).
- The composition of $z_N$ with the map to $H^2_{\text{ét}}(Y, \mathbb{Q}_p(2))^{\text{ord}}$ agrees with a map of Fukaya-Kato for $p \mid N$ up to an Atkin-Lehner involution. They show their map to be equivariant for all Hecke-operators (using dual operators on the right) via a regulator computation. (For $p \nmid N$, it agrees with a map of Lecouturier and J. Wang.)

$p$-adic integrality

We can actually construct a zeta map $z_N$ to $H^2_{\text{ét}}(Y, \mathbb{Z}_p(2))^{\text{ord}}$ after removing the $(\mathbb{Z}/p\mathbb{Z})^\times$-eigenspace for the square of $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \leftrightarrow \mathbb{Z}_p^\times$. 