Reciprocity maps and p-adic L-functions

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Let p be an odd prime.
The goal of this talk is to state a conjecture that relates two sets of objects:

cup products of cyclotomic p-units in Galois cohomology unramified outside p	$\leftrightarrow$	p-adic L-values "mod p" of cusp forms congruent to Eisenstein series

This fits into an Iwasawa and Hida-theoretic conjectural relationship:

A "reciprocity map" applied to a norm compatible sequence of 1 minus p-power roots of 1	$\leftrightarrow$	A two-variable p-adic L-function taken modulo the square of the Eisenstein ideal.

Let $F = \mathbf{Q}(\mu_p)$, $\Delta = \mathrm{Gal}(F/\mathbf{Q})$, $\mu_p = \langle \zeta \rangle$,
$\omega\colon \Delta \to \mu_{p-1}(\mathbf{Z}_p) \subset \mathbf{Z}_p^\times$, $\delta(\zeta) = \zeta^{\omega(\delta)}$ Teichmüller character.

A $\mathbf{Z}_p[\Delta]$-module $\Rightarrow$ eigenspace decompositions

$$A = \bigoplus_{i=0}^{p-2} A^{(i)}, \qquad A^{(i)} = \{a \in A \mid \delta a = \omega(\delta)^i a, \text{ for all } \delta \in \Delta\}$$

and

$$A^{+} = \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{p-3} A^{(i)} \text{ and } A^{-} = \bigoplus_{\substack{i=1 \\ i \text{ odd}}}^{p-2} A^{(i)}.$$

Define

$$\epsilon_i \in \mathbf{Z}_p[\Delta], \qquad \epsilon_i = \frac{1}{p-1} \sum_{\delta \in \Delta} \omega(\delta)^{-i} \delta.$$

Then $\epsilon_i\colon A \to A^{(i)}$.

C_F p-completion of the cyclotomic p-units of F:
$C_F = (1-\zeta)^{\mathbf{Z}_p[\Delta]}$. Set $\eta_i = (1-\zeta)^{\epsilon_{1-i}}$.
$C_F^- = \mu_p$ and $C_F^{(1-i)} = \langle \eta_i \rangle$ for odd i.

A_F p-part of the ideal class group of F.
Herbrand-Ribet: $k \geq 2$ even,

$$A_F^{(1-k)} \neq 0 \Leftrightarrow p \mid \frac{B_k}{k}.$$

We say (p,k) is irregular if $p \mid B_k$ and $2 \leq k \leq p-3$.
Reflection principle:

$$A_F^{(k)} \neq 0 \Rightarrow A_F^{(1-k)} \neq 0, \qquad A_F^{(k)} = 0 \Rightarrow A_F^{(1-k)} \text{ cyclic.}$$

Vandiver's Conjecture: $A_F^+ = 0$. $p <$ 12,000,000 [BCEMS].

$\mathfrak{G}_F$ Galois group of max. unramified outside p extension of F.

$$C_F/pC_F \to H^1(\mathfrak{G}_F, \mu_p) \;\text{ and }\; A_F/pA_F \cong H^2(\mathfrak{G}_F, \mu_p).$$

Cup product

$$H^1(\mathfrak{G}_F, \mu_p) \otimes H^1(\mathfrak{G}_F, \mu_p) \xrightarrow{\cup} H^2(\mathfrak{G}_F, \mu_p^{\otimes 2})$$

sets up a Galois-equivariant pairing

$$(\,\cdot\,,\,\cdot\,)\colon C_F \times C_F \to A_F \otimes \mu_p.$$

For i odd and k even, we have

$$e_{i,k} = (\eta_i, \eta_{k-i}) \in A_F^{(1-k)} \otimes \mu_p.$$

McCallum-S.: Computation of the $e_{i,k}$ up to a single scalar in $\mathbf{Z}/p\mathbf{Z}$ for each irregular (p, k) with $p < 25,000$.
Issue: Scalars could be zero!

Conjecture (McCallum-S.). *The $e_{i,k}$ generate $A_F^{(1-k)} \otimes \mu_p$.*

Theorem (S.). *The conjecture holds for $p < 1000$.*

Examples of the $(e_1\ e_3\ \ldots\ e_{p-2})$ up to scalar:
p = 37, k = 32
(1 26 0 36 1 35 31 34 3 6 2 36 1 0 11 36 11 26)

p = 59, k = 44
(1 45 21 30 14 35 5 0 48 57 7 52 2 11 0 54 24 45 29 38 14 58 27 32 15 0 44 27 32)

p = 67, k = 58
(1 45 38 56 0 47 62 9 29 15 65 26 45 57 0 10 22 41 2 52 38 58 5 20 0 11 29 22 66 2 24 43 65)

p = 101, k = 68
(1 56 40 96 26 63 0 61 81 71 35 92 73 64 6 88 0 0 13 95 37 28 9 66 30 20 40 0 38 75 5 61 45 100 17 17 12 66 72 53 86 31 70 15 48 29 35 89 84 84)

$X_1(p)$ compact modular curve of level p, $C_1(p) = \{\text{cusps}\}$.
$\mathfrak{h} \subset \text{End}_{\mathbf{Z}_p} H_1(X_1(p); \mathbf{Z}_p)$ weight 2 cuspidal Hecke algebra:
generators U_p, T_l ($l \neq p$), $\langle a \rangle$ ($a \in (\mathbf{Z}/p\mathbf{Z})^\times$).

$\mathbb{T} \subset \text{End}_{\mathbf{Z}_p} H_1(X_1(p), C_1(p); \mathbf{Z}_p)$ modular Hecke algebra.

$$H_1(X_1(p); \mathbf{Z}_p) \hookrightarrow H_1(X_1(p), C_1(p); \mathbf{Z}_p)$$

Manin-Drinfeld splitting:

$$s\colon H_1(X_1(p), C_1(p); \mathbf{Q}_p) \twoheadrightarrow H_1(X_1(p); \mathbf{Q}_p),$$

compatible with $\mathbb{T} \twoheadrightarrow \mathfrak{h}$.

$\{a, \infty\}$ class in $H_1(X_1(p), C_1(p); \mathbf{Z}_p)$ of vertical path from $a \in \mathbf{Q}$ to ∞ in upper half plane.
Define $\xi(a)$ for $a \in \mathbf{Q}$ by $\xi(a) = s(\{a, \infty\})$.

Consider ω as a Dirichlet character $(\mathbf{Z}/p\mathbf{Z})^\times \hookrightarrow \mathbf{Z}_p^\times$.

Eisenstein ideal $\mathcal{I}_k \subset \mathfrak{h}$ with character ω^{k-2}: generators $U_p - 1$, $T_l - 1 - l\langle l\rangle$ $(l \neq p)$, $\langle a\rangle - \omega(a)^{k-2}$ $(a \in (\mathbf{Z}/p\mathbf{Z})^\times)$.
Set $\mathfrak{m}_k = p\mathfrak{h} + \mathcal{I}_k$, $\mathfrak{h}_k = \mathfrak{h}_{\mathfrak{m}_k}$.

Let

$$\mathcal{Y}_k = H_1(X_1(p); \mathbf{Z}_p)_{\mathfrak{m}_k} \quad \text{and} \quad \mathcal{Z}_k = s(H_1(X_1(p), C_1(p); \mathbf{Z}_p))_{\mathfrak{m}_k}$$

We have $\mathcal{Y}_k \subset \mathcal{Z}_k$.

complex conjugation: $\mathcal{Z}_k = \mathcal{Z}_k^+ \oplus \mathcal{Z}_k^-$ as $\mathfrak{h}_k$-modules.

$$\mathcal{Z}_k^- = \mathcal{Y}_k^- \quad \text{and} \quad \mathcal{Z}_k^+/\mathcal{Y}_k^+ \cong \mathfrak{h}_k/\mathcal{I}_k.$$

Choice of $\iota\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ yields $\rho\colon G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathfrak{h}_k}\mathcal{Y}_k$.
The following results from work of Ribet, Mazur-Wiles, Wiles, Kurihara, Harder-Pink, Ohta, ...

Theorem. *There exists a homomorphism, canonical up to choice of ι,*

$$\phi_k\colon A_F^{(1-k)} \otimes \mu_p \to \mathcal{Y}_k^+/\mathfrak{m}_k\mathcal{Y}_k^+.$$

that is an isomorphism if $(p, p+1-k)$ is regular.

This is obtained by looking at the map

$$G_{\mathbf{Q}} \to \mathrm{Hom}_{\mathfrak{h}_k}(\mathcal{Y}_k^-, \mathcal{Y}_k^+)$$

induced by ρ.

Remark. Vandiver's conjecture $\Rightarrow A_F^{(1-k)} \otimes \mu_p \cong \mathbf{Z}/p\mathbf{Z}$.

Define $\xi_k\colon \mathbf{Q} \to \mathcal{Z}_k$ as ξ followed by projection to $\mathcal{Z}_k$.
$\kappa\colon \mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$ canonical projection.

For $s \in \mathbf{Z}_p$ and character $\chi\colon (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$, define

$$L_p(\xi_k, \chi) = \sum_{j=1}^{p-1} \chi(j)\xi_k\left(\frac{j}{p}\right) \in \mathcal{Z}_k.$$

In fact, $L_p(\xi_k, \chi) \in \mathcal{Y}_k$.

For i odd, $g_{i,k} =$ image of $L_p(\xi_k, \omega^{i-1})$ in $\mathcal{Y}_k^+/\mathfrak{m}_k\mathcal{Y}_k^+$.

Conjecture. *For each* (p, k) *irregular, there exists* $c_k \in (\mathbf{Z}/p\mathbf{Z})^\times$ *such that*

$$\phi_k(e_{i,k}) = c_k g_{i,k}$$

for all odd i*.*

Eisenstein series of weight 2, level p, character ω^{k-2}:

$$G_{2,\omega^{k-2}} = -\frac{B_{2,\omega^{k-2}}}{2} + 2\sum_{n=1}^{\infty}\left(\sum_{1\le t|n}\omega^{k-2}(t)t\right)q^n.$$

Note: $L_p(\xi_k,\chi)$ determines the analogous p-adic L-values of weight 2, level p cusp forms congruent to $G_{2,\omega^{k-2}}$ modulo p.

Theorem. *Suppose $(p, p+1-k)$ is regular. Then $e_{1,k} \neq 0 \Leftrightarrow g_{1,k} \neq 0 \Leftrightarrow U_p - 1$ generates $\mathcal{I}_k$.*

Remark. Last two conditions are equivalent as (MTT, Kitagawa):

$$L_p(\xi_k, 1) = U_p^{-1}(1 - U_p^{-1})\xi_k(0).$$

Let $G = \mathrm{Gal}(F(\sqrt[p]{C_F})/F)$.
Consider the exact sequence

$$0 \to G \to \mathbf{Z}/p\mathbf{Z}[G]/I_G^2 \to \mathbf{Z}/p\mathbf{Z} \to 0,$$

where I_G is the augmentation ideal in $(\mathbf{Z}/p\mathbf{Z})[G]$.
The coboundary of this twisted by μ_p yields:

$$H^1(\mathfrak{G}_F, \mu_p) \to H^2(\mathfrak{G}_F, \mu_p) \otimes G.$$

I.e., we have a map

$$\Psi_F \colon C_F \to A_F \otimes G.$$

This interpolates the cup products: if $\chi_i \colon G \to \mu_p$ is the Kummer character attached to η_i, then

$$(1 \otimes \chi_i)(\Psi_F(\eta_{k-i})) = (\eta_i, \eta_{k-i})_F.$$

On the other hand, we may view $L_p(\chi_k, \chi)$ as being interpolated by

$$\mathcal{L} = \sum_{j=1}^{p-1} \xi_k \left(\frac{j}{p}\right) \otimes [j] \in \mathcal{Z}_k \otimes \mathbf{Z}_p[(\mathbf{Z}/p\mathbf{Z})^\times].$$

Conjecture extends to compare the “minus parts” of $\Psi_F(1-\zeta)$ and $\mathcal{L} \bmod \mathcal{I}$.

More generally, one can go up the cyclotomic tower (Iwasawa theory) and the modular tower (Hida theory) to compare a reciprocity map on norm compatible sequences of p-units to the two-variable p-adic L-function of Mazur-Kitagawa.