Iwasawa algebras and duality

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Idea of the main result

Goal of Talk (joint with Meng Fai Lim)

Provide an analogue of Poitou-Tate duality which

1. takes place on the level of complexes up to quasi-isomorphism, so in a derived category
2. replaces the Pontryagin dual on coefficients with the Grothendieck dual, i.e., of bounded complexes of finitely generated modules
3. is of Iwasawa cohomology groups up a p-adic Lie extension.

Remarks

- At the level of the ground field (or even up a $\mathbb{Z}_p^r$-extension), such an analogue was derived by Nekovar.
- The more usual case of the Pontryagin dual, but up a p-adic Lie extension, was worked out in detail by Lim.
- Fukaya and Kato proved such an analogue in the case that the coefficients are projective modules (with quite general rings).

Topological modules

Setup

R complete, pro-p commutative local noetherian with finite residue field
Γ compact p-adic Lie group, $\Lambda = R[[\Gamma]]$ noetherian and R-flat
G profinite group, $\chi\colon G \to \Gamma$ continuous, surjective homomorphism

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Notation

Abelian categories $\mathcal{A}$ of Λ-modules:

- Mod_Λ category of Λ-modules
- $\mathcal{C}_{\Lambda,G}$ category of compact Hausdorff $\Lambda[[G]]$-modules
- $\mathrm{Ch}^\star(\mathcal{A})$ with $\star \in \{\varnothing, +, -, b\}$ categories of chain complexes and full subcategories of complexes bounded below, above, and totally

Remarks

a. Objects of $\mathcal{C}_{\Lambda,G}$ are inverse limits of finite $\Lambda[G]$-modules.
b. The topology for the maximal ideal is the unique compact Hausdorff topology on a finitely generated Λ-module.

Complexes of G-cochains

For $T \in \mathcal{C}_{\Lambda,G}$, we have its complex of continuous G-cochains $C(G,T) \in \mathrm{Ch}^+(\mathrm{Mod}_\Lambda)$

More generally, for $T \in \mathrm{Ch}(\mathcal{C}_{\Lambda,G})$, we define $C(G,T) \in \mathrm{Ch}(\mathrm{Mod}_\Lambda)$ as the total direct sum complex of the usual bicomplex, i.e., with ith term

$$C^i(G,T) = \bigoplus_{j \in \mathbb{Z}} C^j(G, T^{i-j}).$$

$H^i(G,T) = H^i(C(G,T))$ is the ith G-hypercohomology group of T.

Remarks

1. As short exact sequences in $\mathcal{C}_{\Lambda,G}$ are split by continuous functions, the $H^i(G,-)$ yield a δ-functor on $\mathrm{Ch}^+(\mathcal{C}_{\Lambda,G})$.
2. The $H^i(G,T)$ depend only on T in $\mathrm{Ch}^+(\mathcal{C}_{\Lambda,G})$ up to quasi-isomorphism, i.e., isomorphism on the cohomology of T.

Complexes of induced modules

Λ° opposite ring of Λ

Remark

We may view Λ as an object of $\mathcal{C}_{\Lambda^\circ,G}$ on which G acts by left multiplication through $\chi\colon G \to \Gamma$ and Λ acts by right multiplication.

Definition

For a complex M of $\Lambda[G]$-modules, M^ι is the complex of $\Lambda^\circ[G]$-modules equal to M over $R[G]$ but $\gamma \in \Gamma$ acts by γ^{-1}.

Definition

For $T \in \mathrm{Ch}(\mathcal{C}_{R,G})$, set $\mathcal{F}_\Gamma(T) = \Lambda^\iota \,\hat{\otimes}_R\, T$.

Remark

Shapiro's lemma provides an isomorphisms in $\mathcal{C}_{\Lambda,G}$:

$$H^i(G, \mathcal{F}_\Gamma(T)) \cong \varprojlim_{\substack{U \trianglelefteq^\circ G \\ \ker\chi \le U}} H^i(U, T),$$

Derived categories

$\mathcal{A}$ an abelian category
Derived category $\mathbf{D}(\mathcal{A})$: localization of homotopy category of $\mathrm{Ch}(\mathcal{A})$ obtained by inverting quasi-isomorphisms.
Bounded derived categories: $\mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$, and $\mathbf{D}^b(\mathcal{A})$.

Remark

Two complexes are isomorphic in $\mathbf{D}(\mathcal{A})$ are isomorphic if and only if they are quasi-isomorphic in $\mathrm{Ch}(\mathcal{A})$.

We have the following standard result:

Theorem

a. *If $\mathcal{A}$ has enough injectives, then every object in $\mathbf{D}^+(\mathcal{A})$ is isomorphic to a bounded below complex of injectives.*
b. *If $\mathcal{A}$ has enough projectives, then every object in $\mathbf{D}^-(\mathcal{A})$ is isomorphic to a bounded above complex of projectives.*

Derived homomorphism groups

Λ, Ω, Σ, Ξ algebras over R
$\mathrm{Mod}_{\Lambda-\Omega}$ category of $\Lambda \otimes_R \Omega^\circ$-modules

We take the following as our definition of derived Λ-module homoms.:

Proposition

Let $A \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Omega})$ and $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$.

1. *If Ω is R-projective and A is bounded above, we have*

$$\mathbf{R}\mathrm{Hom}_\Lambda(A, B) = \mathrm{Hom}_\Lambda(P, B) \in \mathbf{D}(\mathrm{Mod}_{\Omega-\Sigma})$$

   for a complex $P \in \mathrm{Ch}^-(\mathrm{Mod}_{\Lambda-\Omega})$ of Λ-projectives with $P \xrightarrow{\sim} A$.

2. *If Σ is R-flat and B is bounded below, we have*

$$\mathbf{R}\mathrm{Hom}_\Lambda(A, B) = \mathrm{Hom}_\Lambda(A, I) \in \mathbf{D}(\mathrm{Mod}_{\Omega-\Sigma})$$

   for a complex $I \in \mathrm{Ch}^+(\mathrm{Mod}_{\Lambda-\Sigma})$ of Λ-injectives with $B \xrightarrow{\sim} I$.

Remarks

- The two definitions agree in the cases of intersection.
- $H^i(\mathbf{R}\mathrm{Hom}_\Lambda(A, B))$ is the hyper-Ext group $\mathrm{Ext}^i_\Lambda(A, B)$.

Derived tensor products

Proposition

Assume Ω is R-flat. The derived tensor product of $A \in \mathrm{Ch}^-(\mathrm{Mod}_{\Omega-\Lambda})$ and $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$ satisfies

$$A \otimes^{\mathbf{L}}_{\Lambda} B = P \otimes_{\Lambda} B \in \mathbf{D}(\mathrm{Mod}_{\Omega-\Sigma})$$

for $P \in \mathrm{Ch}^-(\mathrm{Mod}_{\Omega-\Lambda})$ of Λ°-flat modules with $P \xrightarrow{\sim} A$.

Remark

$H^{-i}(A \otimes^{\mathbf{L}}_{\Lambda} B)$ is the hyper-Tor group $\mathrm{Tor}_i^{\Lambda}(A, B)$.

Remark

If the algebras are pro-p and Ω is projective in $\mathcal{C}_R$, the derived completed tensor products $A \hat{\otimes}^{\mathbf{L}}_R B \in \mathbf{D}(\mathcal{C}_{\Omega-\Sigma})$ for $A \in \mathrm{Ch}^-(\mathcal{C}_{\Omega-\Lambda})$ and $B \in \mathrm{Ch}(\mathcal{C}_{\Lambda-\Sigma})$ can be computed similarly. One can also incorporate G-actions.

Two useful derived isomorphisms

Lemma A

Let Ξ be R-flat and Σ be R-projective.
Let $A \in \mathrm{Ch}^-(\mathrm{Mod}_{\Omega-\Lambda})$, $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$, and $C \in \mathrm{Ch}^+(\mathrm{Mod}_{\Omega-\Xi})$.
Then we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_{\Sigma-\Xi})$ given by

$$\mathbf{R}\mathrm{Hom}_\Omega(A \otimes^{\mathbf{L}}_\Lambda B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(B, \mathbf{R}\mathrm{Hom}_\Omega(A, C)).$$

Lemma B

Let Ω be R-flat and Ξ be R-projective.
Let $A \in \mathrm{Ch}^b(\mathrm{Mod}_{\Omega-\Sigma})$, $B \in \mathrm{Ch}^-(\mathrm{Mod}_{\Xi-\Lambda})$, and $C \in \mathrm{Ch}^+(\mathrm{Mod}_{\Sigma-\Lambda})$.
Suppose that

1. *the terms of A are Σ°-flat and*
2. *the terms of B are finitely presented and projective over Λ°.*

Then we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_{\Omega-\Xi})$ given by

$$A \otimes^{\mathbf{L}}_\Sigma \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(B, A \otimes^{\mathbf{L}}_\Sigma C).$$

Two useful derived isomorphisms

Lemma A

Let Ξ *be* R*-flat and* Σ *be* R*-projective.*
Let $A \in \mathrm{Ch}^{-}(\mathrm{Mod}_{\Omega-\Lambda})$, $B \in \mathrm{Ch}(\mathrm{Mod}_{\Lambda-\Sigma})$, *and* $C \in \mathrm{Ch}^{+}(\mathrm{Mod}_{\Omega-\Xi})$.
Then we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_{\Sigma-\Xi})$ *given by*

$$\mathbf{R}\mathrm{Hom}_{\Omega}(A \otimes^{\mathbf{L}}_{\Lambda} B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda}(B, \mathbf{R}\mathrm{Hom}_{\Omega}(A, C)).$$

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Let Ω *be* R*-flat and* Ξ *be* R*-projective.*
Let $A \in \mathrm{Ch}^{b}(\mathrm{Mod}_{\Omega-\Sigma})$, $B \in \mathrm{Ch}^{-}(\mathrm{Mod}_{\Xi-\Lambda})$, *and* $C \in \mathrm{Ch}^{+}(\mathrm{Mod}_{\Sigma-\Lambda})$.
Suppose that

1. *the terms of* A *are* Σ°*-flat and*
2. *the terms of* B *are finitely presented and projective over* Λ°.

Then we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_{\Omega-\Xi})$ *given by*

$$A \otimes^{\mathbf{L}}_{\Sigma} \mathbf{R}\mathrm{Hom}_{\Lambda^{\circ}}(B, C) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^{\circ}}(B, A \otimes^{\mathbf{L}}_{\Sigma} C).$$

Spectral sequences

Let T be a complex in $\mathcal{C}_{R,G}$.
$\mathbf{R}\Gamma(G, \mathcal{F}_\Gamma(T))$ is $C(G, \mathcal{F}_\Gamma(T))$ viewed as an object of $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

Theorem

Let $\Gamma_0 \trianglelefteq \Gamma$ be closed, $\Gamma' = \Gamma/\Gamma_0$, and $\Lambda' = R[[\Gamma']]$. Consider

1. *G has finite cohomological dimension and*
2. *Γ_0 contains no elements of order p.*

If (1) holds and T is bounded above, (2) holds and T is bounded below, or both (1) and (2) hold, then we have an isomorphism

$$\Lambda' \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\Gamma(G, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\Gamma(G, \mathcal{F}_{\Gamma'}(T))$$

in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$.

Remark

In other words, we have a descent spectral sequence

$$\mathrm{Tor}^{\Lambda}_{-i}(\Lambda', H^j(G, \mathcal{F}_\Gamma(T))) \Rightarrow H^{i+j}(G, \mathcal{F}_{\Gamma'}(T)).$$

Cones and shifts

Definition

For $i \in \mathbb{Z}$, the ith *shift* of a complex A is the complex $A[i]$ with

$$A[i]^j = A^{i+j}, \qquad d^j_{A[i]} = (-1)^i d^{i+j}_A.$$

Definition

The *cone* of a map $f\colon A \to B$ of complexes is the complex

$$\mathrm{Cone}(f)^i = A^{i+1} \oplus B^i, \qquad d^i_{\mathrm{Cone}(f)}(a,b) = (d^i_A(a), f(a) - d^i_B(b)).$$

Lemma

The cone of $f\colon A \to B$ fits into an exact sequence of complexes

$$0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$$

giving rise to a long exact sequence

$$H^i(B) \to H^i(\mathrm{Cone}(f)) \to H^{i+1}(A) \xrightarrow{f^{i+1}} H^{i+1}(B).$$

Arithmetic setup

Notation

F global field of characteristic not p
S finite set of primes of F including any over p and any real places
G_S Galois group of maximal S-ramified extension of F
G_v absolute Galois group of completion F_v at any $v \in S$
$\Gamma = \mathrm{Gal}(F_\infty/F)$ with F_∞/F an S-ramified p-adic Lie extension
T bounded complex in $\mathcal{C}_{R,G_{F,S}}$ of finitely generated R-modules

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Definition

The *compactly supported G_S-cochains* of T are

$$C_c(G_S,T) = \mathrm{Cone}\Big(C(G_S,T) \xrightarrow{\sum \mathrm{Res}_v} \bigoplus_{v\in S} C(G_v,T)\Big)[-1],$$

where we use Tate cochain complexes for archimedean v.

The cone provides an exact triangle

$$\mathbf{R}\Gamma_c(G_S,T) \to \mathbf{R}\Gamma(G_S,T) \to \bigoplus_{v\in S}\mathbf{R}\Gamma(G_v,T) \to \mathbf{R}\Gamma_c(G_S,T)[1]$$

which yields the long exact sequence on cohomology.

Finite generation of cohomology

Proposition (Descent spectral sequences)

Let $\Gamma' = \mathrm{Gal}(F'_\infty/F)$ be a quotient of Γ by a closed subgroup. If p is odd or all real places of F'_∞ split completely in F_∞, then we have an isomorphism of exact triangles in $\mathbf{D}(\mathrm{Mod}_{\Lambda'})$:

$$\begin{array}{ccc}
\Lambda' \otimes_\Lambda \mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\Gamma_c(G_S, \mathcal{F}_{\Gamma'}(T)) \\
\downarrow & & \downarrow \\
\Lambda' \otimes_\Lambda \mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\Gamma(G_S, \mathcal{F}_{\Gamma'}(T)) \\
\downarrow & & \downarrow \\
\bigoplus_{v\in S} \Lambda' \otimes_\Lambda \mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, \mathcal{F}_{\Gamma'}(T)) \\
\downarrow & & \downarrow \\
\Lambda' \otimes_\Lambda \mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T))[1] & \xrightarrow{\sim} & \mathbf{R}\Gamma_c(G_S, \mathcal{F}_{\Gamma'}(T))[1]
\end{array}$$

Theorem

The ith cohomology groups $H^i(G_S, \mathcal{F}_\Gamma(T))$, $H^i_c(G_S, \mathcal{F}_\Gamma(T))$, and $H^i(G_v, \mathcal{F}_\Gamma(T))$ are all finitely generated Λ-modules.

Modified Čech complex

Let $\mathfrak{m}$ denote the maximal ideal and d the Krull dimension of R.

Let $x_1, \dots, x_d \in \mathfrak{m}$ be such that $R/(x_1, \dots, x_d)$ has Krull dimension 0.

Definition

The *modified Čech complex* for R is the complex

$$C_R = \Big[R \to \bigoplus_i R_{x_i} \to \bigoplus_{i<j} R_{x_i x_j} \to \cdots \to R_{x_1 \dots x_d}\Big][d]$$

with appropriate signs (i.e., $(-1)^{s+1} \colon R_{x_{i_1} \cdots \widehat{x_{i_s}} \cdots x_{i_t}} \to R_{x_{i_1} \cdots x_{i_t}}$).

Remark

The complex $J_R = \operatorname{Hom}_R(C_R, R^\vee)$ is of R-injectives and is quasi-isomorphic to a bounded complex of finitely generated R-modules.

Grothendieck duality

Example

If $R = \mathbb{Z}_p$ and $x_1 = p$, then

$$C_{\mathbb{Z}_p} = [\mathbb{Z}_p \to \mathbb{Q}_p][1] \quad \text{and} \quad J_{\mathbb{Z}_p} = [\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p] \simeq \mathbb{Z}_p.$$

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Definition

The *dualizing complex* ω_R is the object represented by J_R in the derived category $\mathbf{D}_{R-\mathrm{ft}}(\mathrm{Mod}_R)$ of complexes that are quasi-isomorphic to bounded complexes of finitely generated R-modules.

Theorem (Grothendieck duality)

For $T \in \mathbf{D}^b_{R-\mathrm{ft}}(\mathrm{Mod}_R)$, there exists a canonical quasi-isomorphism

$$T \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(T, \omega_R), \omega_R).$$

Remark

If R is Gorenstein, then ω_R is quasi-isomorphic to R.

Nekovář's version of Poitou-Tate duality

Notation

T bounded complex in $\mathcal{C}_{R,G_S}$ of finitely generated R-modules
T^* quasi-isomorphic subcomplex of $\mathrm{Hom}_R(T, J_R)$ in $\mathrm{Ch}^b(\mathcal{C}^{R-\mathrm{ft}}_{R,G_S})$

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$T \otimes_R T^*(1) \to J_R(1)$ on the total tensor product yields a cup product. The adjoint maps then provide morphisms as in the following.

Theorem (Nekovář)

There are natural, compatible isomorphisms in $\mathbf{D}(\mathrm{Mod}_R)$ *given by*

$$\mathbf{R}\Gamma(G_v, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma(G_v, T^*(1)), \omega_R)[-2],$$
$$\mathbf{R}\Gamma(G_S, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S, T^*(1)), \omega_R)[-3],$$
$$\mathbf{R}\Gamma_c(G_S, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma(G_S, T^*(1)), \omega_R)[-3].$$

Remark

The theorem yields the hypercohomology spectral sequence

$$\mathrm{Ext}_R^i(H^j(G_S, T^*(1)), \omega_R) \Rightarrow H_c^{3-i-j}(G_S, T).$$

Duality for complexes of induced modules

Would like a Nekovář-type duality up the Iwasawa tower between co-homology groups with coefficients in $\mathcal{F}_\Gamma(T)$ and $\mathcal{F}_\Gamma(T^*)^\iota(1)$.

Problem

The R-algebra Λ can be noncommutative, and we do not know if it has a dualizing complex of bimodules analogous to ω_R.

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$\Lambda \otimes_R \omega_R$ plays the role of ω_R for cohomology of induced modules.

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Proposition

We have a natural isomorphism

$$\mathcal{F}_\Gamma(T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathcal{F}_\Gamma(T^*)^\iota, \Lambda \otimes_R \omega_R)$$

in $\mathbf{D}^b(\mathrm{Mod}_{\Lambda[G_S]})$. *In fact, this can be seen in the bounded derived category of the ind-category of the full subcategory of finitely generated modules in* $\mathcal{C}_{\Lambda,G_S}$.

Main result

Theorem (Lim-S.)

We have a natural isomorphism of exact triangles in $\mathbf{D}(\mathrm{Mod}_\Lambda)$*:*

$$\begin{array}{ccc}
\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} \mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T)) & \xrightarrow{\sim} & \bigoplus_{v \in S} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_v, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T))[1] & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-2]
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\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T))[1] & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-2]
\end{array}$$

Remark

Any 2 of the first 3 isomorphisms implies the third; the second and the third for nonarchimedean summands are easiest. At real places, the third involves additional technicalities (in particular, q-projective resolutions) to deal with the cohomologically unbounded complexes.

Consequences

Our results plus Poitou-Tate duality (in the sense of Lim's work) yield:

Corollary

Have convergent spectral sequences of finitely generated Λ/Λ°-modules:

$$\mathrm{Ext}^i_{\Lambda^\circ}(H^j(H_S, T \otimes_R C_R)^\vee, \Lambda \otimes_R J_R) \Rightarrow H^{i+j}(G_S, \mathcal{F}_\Gamma(T))$$
$$\mathrm{Ext}^i_{\Lambda}(H^j(G_S, \mathcal{F}_\Gamma(T)), \Lambda \otimes_R J_R) \Rightarrow H^{i+j}(H_S, T \otimes_R C_R)^\vee$$

with H_S the Galois group of the maximal S-ramified extension of F_∞.

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with H_S the Galois group of the maximal S-ramified extension of F_{∞}.

Remark

If R is Gorenstein and T is R-flat, the first spectral sequence becomes

$$\mathrm{Ext}^i_{\Lambda^{\circ}}(H^j(H_S, T \otimes_R R^{\vee})^{\vee}, \Lambda) \Rightarrow H^{i+j}(G_S, \mathcal{F}_{\Gamma}(T)),$$

which generalizes the spectral sequence of Jannsen for $R = \mathbb{Z}_p$ that is useful in computing Iwasawa adjoints.

An application

Setup

$\mathfrak{h}$ Hida's ordinary cuspidal $\mathbb{Z}_p$-Hecke algebra of level Np^∞, $p \geq 5$, $p \nmid N$
$\mathcal{S}$ space of ordinary "Λ-adic" cusp forms acted on by $\mathfrak{h}$
$\Lambda = \mathfrak{h}[[\Gamma]]$ for $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with $\mathbb{Q}_\infty/\mathbb{Q}$ unramified outside $S = \{v \mid Np\infty\}$ and containing $\mathbb{Q}(\mu_{Np^\infty})$
$\mathcal{H}$ inverse limit of $H^1_{\text{ét}}(X_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\mathrm{ord}}$

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Remarks (Ohta, Fukaya-Kato)

- $\mathcal{H} \cong \mathcal{S} \oplus \mathfrak{h}$ as $\mathfrak{h}$-modules via adjoint Hecke action.
- $\mathcal{H} \times \mathcal{H} \to \mathcal{S}(1)$ twisted Poincaré duality pairing that is perfect, $\mathfrak{h}$-bilinear, and G_S-equivariant for $\sigma \in G_S$ acting on $\mathcal{S}$ by $\langle \chi(\sigma) \rangle$ with χ the cyclotomic character to $\mathbb{Z}_{p,N}^\times = \varprojlim (\mathbb{Z}/Np^r\mathbb{Z})^\times$.
- $\mathfrak{h}$ is Cohen-Macaulay with dualizing module $\mathcal{S}$.

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Remarks (Ohta, Fukaya-Kato)

- $\mathcal{H} \cong \mathcal{S} \oplus \mathfrak{h}$ as $\mathfrak{h}$-modules via adjoint Hecke action.
- $\mathcal{H} \times \mathcal{H} \to \mathcal{S}(1)$ twisted Poincaré duality pairing that is perfect, $\mathfrak{h}$-bilinear, and G_S-equivariant for $\sigma \in G_S$ acting on $\mathcal{S}$ by $\langle \chi(\sigma) \rangle$ with χ the cyclotomic character to $\mathbb{Z}_{p,N}^\times = \varprojlim (\mathbb{Z}/Np^r\mathbb{Z})^\times$.
- $\mathfrak{h}$ is Cohen-Macaulay with dualizing module $\mathcal{S}$.

In $\mathbf{D}(\mathrm{Mod}_\Lambda)$, it follows that we have an isomorphism

$$\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(\mathcal{H})) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(\mathcal{H}))^\iota, \Lambda \otimes_{\mathfrak{h}} \mathcal{S})\langle \chi \rangle[-3]$$

where $\langle \chi \rangle$ indicates the action of $\gamma \in \Gamma$ is twisted by $\langle \chi(\gamma) \rangle$.

Reduction to torsion-free coefficient rings

$\mathcal{R}$ a complete commutative local noetherian ring with quotient R
$\Omega = \mathcal{R}[[\Gamma]]$ and $\mathfrak{m}_{\mathcal{R}}$ the maximal ideal of $\mathcal{R}$
$d = \dim_k \mathfrak{m}_{\mathcal{R}}/\mathfrak{m}_{\mathcal{R}}^2 - \dim_k \mathfrak{m}/\mathfrak{m}^2$

Proposition

We have the following commutative diagram in $\mathbf{D}(\mathrm{Mod}_\Sigma)$*:*

$$\begin{array}{ccc} \mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3] \\ \downarrow \wr & & \downarrow \wr \\ \mathbf{R}\Gamma(G_S, \Omega^\iota \otimes_{\mathcal{R}} T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Omega^\circ}(\mathbf{R}\Gamma_c(G_S, \Omega \otimes_{\mathcal{R}} T^*[-d](1)), \Omega \otimes_{\mathcal{R}} \omega_{\mathcal{R}})[-3]. \end{array}$$

Remarks

1. $\mathbf{R}\mathrm{Hom}_R(T, \omega_{\mathcal{R}}) \cong \mathbf{R}\mathrm{Hom}_R(T, \omega_R)[-d]$
2. $\Omega^\iota \otimes_{\mathcal{R}} T \cong \Lambda^\iota \otimes_R T = \mathcal{F}_\Gamma(T)$

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Remark

R is a quotient of a polynomial ring in $\dim_k \mathfrak{m}/\mathfrak{m}^2$ variables over the Witt vectors of $k = R/\mathfrak{m}$, so we may assume R is regular and $\mathbb{Z}_p$-flat.

Idea of proof for torsion-free coefficient rings

To show:

$$\mathbf{R}\Gamma(G_S, \mathcal{F}_\Gamma(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda \otimes_R \omega_R)[-3].$$

Know:

$$\mathbf{R}\Gamma(G_S, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S, T^*(1)), \omega_R)[-3].$$

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Idea

Induction working modulo powers of the augmentation ideal I of Λ.

Note that $\Lambda = \varprojlim \Lambda/I^n$.
Moreover, $\Lambda/I \cong R$, and I^n/I^{n+1} is a finitely generated R-module with an R-flat resolution

$$[I^{n+1} \to I^n] \xrightarrow{\sim} I^n/I^{n+1}.$$

Finally, note that I^n/I^{n+1} (unlike Λ) has a trivial G_S-action.

Inductive step

For $0 \leq m < n$, set $\mathcal{F}_{I^m/I^n}(T) = [I^n \to I^m] \otimes_\Lambda \mathcal{F}_\Gamma(T)$.

We have a commutative diagram of exact triangles:

$$\begin{array}{ccc}
\mathbf{R}\Gamma(G_S, \mathcal{F}_{I^n/I^{n+1}}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_S, \mathcal{F}_{\Lambda/I^{n+1}}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
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\mathbf{R}\Gamma(G_S, \mathcal{F}_{\Lambda/I^n}(T)) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_S, \mathcal{F}_{I^n/I^{n+1}}(T))[1] & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-2]
\end{array}$$

When $n = 1$, the third horizontal map becomes Nekovář's theorem after applying the descent spectral sequence.

By induction, the second horizontal map is an isomorphism if we can show that the first is (by the five lemma).

End of the proof

Using the flatness of $[I^{n+1} \to I^n]$ and the triviality of the G_S-action on I^n/I^{n+1}, we obtain a natural commutative diagram in $\mathbf{D}(\mathrm{Mod}_R)$:

$$\begin{array}{ccc}
I^n/I^{n+1} \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma(G_S,T) & \xrightarrow{\sim} & I^n/I^{n+1} \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S,T^*(1)),\omega_R)[-3] \\
\downarrow \wr & & \text{Lemma B} \downarrow \wr \\
\mathbf{R}\Gamma(G_S, I^n/I^{n+1} \hat{\otimes}_R^{\mathbf{L}} T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S,T^*(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R)[-3] \\
 & & \text{Descent} \downarrow \wr \\
\downarrow \wr & & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S,\mathcal{F}_\Gamma(T^*)^\iota(1)) \otimes_\Lambda^{\mathbf{L}} R, I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R)[-3] \\
 & & \text{Lemma A} \downarrow \wr \\
\mathbf{R}\Gamma(G_S,\mathcal{F}_{I^n/I^{n+1}}(T)) & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S,\mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R)[-3].
\end{array}$$

The upper isomorphism follows from Nekovář's theorem.

End of the proof

Using the flatness of $[I^{n+1} \to I^n]$ and the triviality of the G_S-action on I^n/I^{n+1}, we obtain a natural commutative diagram in $\mathbf{D}(\mathrm{Mod}_R)$:

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I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(G_S, T) & \xrightarrow{\sim} & I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S, T^*(1)), \omega_R)[-3] \\
\downarrow \wr & & \text{Lemma B} \downarrow \wr \\
\mathbf{R}\Gamma(G_S, I^n/I^{n+1} \hat{\otimes}^{\mathbf{L}}_R T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S, T^*(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
\big\downarrow \wr & & \text{Descent} \downarrow \wr \\
 & & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)) \otimes^{\mathbf{L}}_\Lambda R, I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3] \\
 & & \text{Lemma A} \downarrow \wr \\
\mathbf{R}\Gamma(G_S, \mathcal{F}_{I^n/I^{n+1}}(T)) & \to & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R)[-3].
\end{array}$$

The upper isomorphism follows from Nekovář's theorem.

To finish the proof, one then needs only to pass to the inverse limit over n of the isomorphisms

$$\mathbf{R}\Gamma(G_S, \mathcal{F}_{\Lambda/I^n}(T)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(\mathbf{R}\Gamma_c(G_S, \mathcal{F}_\Gamma(T^*)^\iota(1)), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R)[-3].$$

For this, we resolve the compactly supported cochains by a complex of finitely generated projective Λ°-modules, which allows us to pass the inverse limit through the homomorphism complex to the second term.