Symplectic Manifolds

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July 2023

Symplectic Linear Algebra

- A *symplectic vector space* is a finite-dimensional vector space V equipped with a nondegenerate alternating bilinear form ω.
- This form gives an isomorphism from V to V^* given by $v \to \omega(-, v)$.
- These are always even dimensional!

Subspaces

- A subspace W of a symplectic vector space (V, ω) is called *isotropic* if $\omega|_W = 0$ i.e. $W \subset W^{\omega\perp}$.
- W is called *Lagrangian* if W is isotropic of maximal dimension i.e. $W = W^{\omega\perp}$.
- $dim(W) + dim(W^{\omega\perp}) = dim(V)$ for *any* subspace W

Symplectic Manifolds

- A **** manifold M and differential 2-form ω are called a *symplectic manifold* if ω is closed ($d\omega = 0$) and $\omega|_p$ is nondegenerate (hence symplectic) for all $p \in M$.
- We will allow M to be real C^∞ or complex nonsingular algebraic.
- ω gives a canonical bijection between vector fields on M and differential 1-forms on M given by $X \to \omega(-, X)$.
- These are always even-dimensional!
- These are always orientable!

Examples

- $(\mathbb{R}^{2n}, \omega)$ with $\omega = \sum_{i=1}^{n} dx_i dy_i$
- S^2 with any nonvanishing volume form
- Let G be a Lie/algebraic group acting on the dual algebra $\mathfrak{g}^*$ via the coadjoint action. Any orbit of this action is a symplectic manifold with the Kirillov-Kostant-Souriau form.

Classical Example

- If M is any n-dimensional manifold, then T^*M, the cotangent bundle, is canonically a symplectic manifold with the form $\sum_{i=1}^{n} dp_i dq_i$, where p_i is a set of local coordinates on M, and q_i are the corresponding extra coordinates on T^*M.
- If M represents a set of position coordinates for a physical object, then T^*M is the "phase space" tracking position and momentum. This is the basis for Hamiltonian mechanics.
- The Heisenberg uncertainty principle says the position and momentum of quantum particles cannot be observed simultaneously. So, the smallest observables in the corresponding phase space are the Lagrangian submanifolds i.e. the "quantum points"

Poisson Structures

A *Poisson structure* on a manifold M (generally a variety) is a k-bilinear map $\{-,-\} : \mathcal{O}_M \times \mathcal{O}_M \to \mathcal{O}_M$ such that, for every open $U \subset M$, $f \in \mathcal{O}_M(U)$

- $\{-,-\}$ is a Lie bracket on $\mathcal{O}_M(U)$
- $\{f,-\}$ is a derivation of $\mathcal{O}_M(U)$ i.e. $\{f, gh\} = \{f,g\}h + g\{f,h\}$

E.G. If $\mathcal{O}_M(U) = Mat_n(\mathbb{R})$, then the standard Lie bracket $[A,B] = AB - BA$ is a Poisson bracket on this level.
Morally, a Poisson manifold is a symplectic manifold where the form is allowed to have some degeneracy

Vector Fields and 1-forms

- Recall that $X \to \omega(-, X)$ is a bijection from vector fields to 1-forms
- Given a 1-form η, denote the associated vector field as X_η.
- If $\eta = df$ for some smooth function f, denote $X_f = X_{df} = X_\eta$

Symplectic Vector Fields

- A vector field X is called *symplectic* if $\omega(-, X)$ is closed
- Reason: Symplectic vector fields preserve ω i.e. $L_X\omega = 0$
- Proof: $L_X\omega = i_X d\omega + d(i_X\omega) = 0 + 0 = 0$
- X_f is always symplectic, since $\omega(-, X_f) = df$ is exact

Lie Morphism

Theorem:

- If (M, ω) is symplectic, then $\{f, g\} = \omega(X_g, X_f)$ is a Poisson structure on M.
- In fact, $f \to X_f$ is a Lie algebra morphism; $X_{\{f,g\}} = [X_f, X_g]$

Hamiltonian Vector Fields

- A vector field X is called *Hamiltonian* if $\omega(-, X) = dH$ for some smooth H.
- H is often called the *Hamiltonian function*
- Fact: H is constant on any integral curve of X_H
- This reflects that physical objects travel along paths that conserve their energy/momentum
- We need to consider Lie group actions if we want multiple degrees of symmetry/conservation

Group Actions

- Let G be a Lie/Linear Algebraic group acting on M *symplectically*, meaning $g^*\omega = \omega$ for all $g \in G$. We denote by $\mathfrak{g}$ the Lie algebra of G.
- For any $a \in \mathfrak{g}$, there is an associated vector field X_a on M corresponding to the infinitesimal motion of the group action.
- Fact: If G acts symplectically, X_a is always symplectic i.e. $L_{X_a}\omega = 0$.
- We would like it if X_a were always Hamiltonian so that we can turn symmetries of motion into conserved quantities.
- Note that in the C^∞ case, symplectic vector fields are *locally* Hamiltonian since manifolds are locally Euclidean

Desired Function

- We are essentially hoping for a function $H : M \times \mathfrak{g} \to \mathbb{R}$ sending (x, a) to $H_a(x)$, where H_a is the Hamiltonian for a.
- This would ideally be linear in the first variable. This would therefore give us a map $\mu : M \to \mathfrak{g}^*$ given by $x \to H(x, -)$.
- It would also be nice if $a \to H_a$ were a Lie morphism i.e. $\{H_a, H_b\} = H_{[a,b]}$

Hamiltonian Actions

A symplectic action of G on M is called *Hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{g}^*$ such that:

1. For any $a \in \mathfrak{g}$, the function $H_a(x) = \mu(x)(a)$ is the Hamiltonian for X_a i.e. $X_{H_a} = X_a$.
2. For any $a, b \in \mathfrak{g}$, $\{H_a, H_b\} = H_{[a,b]}$.
3. μ is G-equivariant

Note that such μ need not be unique, but we will assume μ fixed in the future.

Example

- Take $M = \mathbb{C}^2$ with coordinates (p, q) and so $\omega = dpdq$. $G = SL_2(\mathbb{C})$ acts on M symplectically.
- The generators of $\mathfrak{g}$ map to the corresponding vector fields

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to q\frac{\partial}{\partial p}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to p\frac{\partial}{\partial q}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$$

- This action can likely be made Hamiltonian since $H^1(M) = 0$

Example Continued

- This action is in fact Hamiltonian. The given vector fields come from smooth functions as follows:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to q\frac{\partial}{\partial p} = X_{q^2/2}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to p\frac{\partial}{\partial q} = X_{-p^2/2}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} = X_{pq}$$

- The corresponding moment map is $\mu : (p, q) \to 1/2 \begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}$

Motivation

- We will show that under nice conditions, certain quotients exist in the symplectic category
- This reflects eliminating symmetries or conserved quantities to see a physical system's "true" dynamics
- The general setup is that we have a symplectic manifold (M, ω) along with a Hamiltonian acting Lie/linear algebraic group (G, μ).
- We should not naively try to look at the orbit space M/G. For one thing, this isn't even dimensional in general!

Level Set

- We will consider certain subspaces of M. Let $p \in \mathfrak{g}^*$ be a fixed point of the coadjoint action. Often, one takes $p = 0$.
- By equivariance of μ, $\mu^{-1}(p)$ is closed under the G action
- Fact: If G acts freely and properly on $\mu^{-1}(p)$, then $\mu^{-1}(p)$ is a coisotropic submanifold of M.
- In fact, $\mu^{-1}(p)$ is foliated by orbits

Quotient Manifold

- Now, restrict to the smooth C^∞ symplectic category.
- Near any $x \in \mu^{-1}(p)$, M looks like a bunch of parallel G orbits bound together. We want to try and quotient out these extra dimensions by passing to the orbit space.
- In general, $\mu^{-1}(p)/G$ need not be a manifold. But in the case that G acts freely and properly, it will be a manifold!
- The quotient will also inherit the symplectic form from $\mu^{-1}(p)$, and this form will still be nondegenerate

Hamiltonian Reduction

To summarize, we have this variant of the Marsden-Weinsten theorem:
Let M be a C^∞ symplectic manifold with a proper Hamiltonian action of a real Lie group G. If $p \in \mathfrak{g}^*$ is fixed by G and G acts properly and freely on $\mu^{-1}(p)$, then $\mu^{-1}(p)/G$ has the canonical structure of a symplectic manifold.

- Even if p isn't fixed, one can alter the hypotheses to consider μ^{-1} of an entire orbit

Hamiltonian Reduction Special Case

- Suppose a Lie group G acts freely and properly on a manifold X. One can show that this lifts to a free and proper action on $M = T^*X$. The action on T^*X is actually Hamiltonian with $\mu(x, \lambda)(a) = \lambda(X_a(x))$
- By the theorem, $\mu^{-1}(0)/G$ is a symplectic manifold. It is in fact symplectomorphic to $T^*(X/G)$
- This result holds for smooth algebraic varieties as well as C^∞ manifolds.

Non-free Action

- Consider this previous example in the algebraic case but with a possibly non-free action
- If G acts non-freely, then $\mu^{-1}(0)/G$ may not be smooth. It would be better to study what structure GIT quotients have.
- Let $\chi : G \to k^{\times}$ be a character, and recall the *twisted GIT quotient* $\mathcal{M}_{\chi} = (\mu^{-1}(0))//_{\chi}G$
- These come with projective morphisms $\pi : \mathcal{M}_{\chi} \to \mathcal{M}_0$

Stable Points

- Recall the special class of χ-semistable points, $(\mu^{-1}(0))^s$, called the *stable* points. One important condition is that all G_x are finite.
- When this set is nonempty, it is open in both $(\mu^{-1}(0))^{ss}$ and $\mu^{-1}(0)$
- Denote the quotient $\mathcal{M}^s_\chi := (\mu^{-1}(0))^s//_\chi G$
- Generally just better behaved and easier to check many conditions on these - e.g. freeness of G action

GIT

- Fact: For any χ, $\mathcal{M}_\chi$ has a Poisson structure inherited from T^*X, and the projective map $\pi : \mathcal{M}_\chi \to \mathcal{M}_0$ is Poisson
- Moreover, when $\mathcal{M}_\chi^s$ is smooth, it is in fact canonically a symplectic manifold, with the Poisson structure above coinciding with the one induced by the symplectic form.
- $T^*(X^s//G)$ is an open (possibly empty) subset of $\mathcal{M}_\chi^s$

- Kirillov “Quiver Representations and Quiver Varieties”
- Chriss and Ginzburg “Representation Theory and Complex Geometry”
- McDuff and Salamon “Introduction to Symplectic Topology”
- Lee “Introduction to Smooth Manifolds”
- Wikipedia “Poisson Manifold”