A Mixed Approach to Poincaré Lemma

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The Poincaré lemma is essential to computing de Rham cohomology of smooth manifolds, as it gives a sort of base case to reduce to when using the Mayer-Vietoris sequence. The lemma essentially shows that de Rham cohomology is homotopy invariant, and hence, can be computed readily via homotopic techniques. Unfortunately, the proofs given in many standard texts are either "too geometric" or "too algebraic". The "too geometric" proofs prefer to avoid the language of cochain complexes and chain homotopies, and so miss some of the generality of the theorem. The "too algebraic" proofs ignore the geometric intuition and instead write down the explicit homotopy operator, apparently obtained via divine revelation. This writeup is an attempt to combine the most useful techniques from both styles to produce an organized and motivated proof of the Poincaré lemma. This proof tries to emphasize how geometric ideas are being converted into algebraic ones.

Lemma 1. *Any star-shaped domain in $\mathbb{R}^n$ (in particular, $\mathbb{R}^n$ itself) is homotopy equivalent to a single point*

Proof. Take a central point x_0 in our domain D. Then the homotopy $\phi : D\times[0,1] \to D$, $\phi(x,t) = (1-t)x+tx_0$ is a smooth deformation retract of D onto $\{x_0\}$. Note that the image of ϕ is always contained in D since D is star-shaped, and this is simply a radial contraction onto the central point. □

Lemma 2. *Let D and C be smooth manifolds without boundary (although this proof can be modified to work if there is boundary). If $\phi : D \times [0,1] \to C$ is a smooth homotopy between the smooth ϕ_0 and ϕ_1, then the pullback maps of de Rham complexes $\phi_0^*, \phi_1^* : \Omega^*(C) \to \Omega^*(D)$ are chain homotopic.*

Proof. Define $e_t : D \to D\times[0,1]$, $x \to (x,t)$. Then $\phi_t = \phi\circ e_t$. On cochain complexes, we have the composites $\phi_t^* = e_t^*\circ\phi^*$, with $\phi^* : \Omega^*(C) \to \Omega^*(D\times[0,1])$, and $e_t^* : \Omega^*(D\times[0,1]) \to D$. It is sufficient then to find a chain homotopy from e_0^* to e_1^*, since precomposing with ϕ^* will give a chain homotopy from ϕ_0^* to ϕ_1^*. It is much easier to believe that e_0^* and e_1^* should be chain homotopic, since e_0 and e_1 are just different embeddings of D.

Recall that a chain homotopy from $\Omega^*(D \times [0,1])$ to $\Omega^*(D)$ has component maps that reduce degree by 1. So, we need a tool that lets us *reduce* the degree of a differential form, and a tool that lets us eliminate the last coordinate to change manifolds. Our first problem is solved by the interior product of forms i_X; if we imagine a differential p-form as a smooth function of p different vector fields, then $i_X\omega$ has its first argument fixed as the vector field X, and so is only a $p-1$ form. To answer our second problem, we could use any e_t^*, but no value of t is preferred. So, we will simply integrate across t, essentially removing our extraneous coordinate. Integrating from 0 to 1 is as easy as it gets! We are, in a sense, taking an average value across the coordinate that we wish to ignore.

Formally, on $D \times [0,1]$, we have the vector field $X = (0, \partial/\partial t')$ that is only a differential in the last t coordinate. This vector field is chosen to encode that this is the coordinate we wish to eliminate. Denote i_X to be the interior product of forms with respect to this field. Then, define $h : \Omega^p(D \times [0,1]) \to \Omega^{p-1}(D)$, $h\omega = \int_0^1 e_t^*(i_X(\omega))dt$ (if $p = 0$, we take the zero map). $h\omega$ is well-defined by thinking of the integral as a limit of sums. We show that this map h, which is clearly $\mathbb{R}$-linear, is the chain homotopy that we seek.

First, d commutes with integrals via working in local coordinates, so $dh\omega = \int_0^1 de_t^*(i_X(\omega))dt$. Therefore, $(h_{p+1}d_{D\times[0,1],p} + d_{D,p-1}h_p)(\omega) = \int_0^1 e_t^*(i_X(d\omega)) + de_t^*(i_X(\omega))dt$. d commutes with pullbacks, so this is

$\int_0^1 e_t^*(i_X d + di_X(\omega))dt = \int_0^1 e_t^* \mathcal{L}_X(\omega)dt$, where $\mathcal{L}_X$ is the Lie derivative with respect to X.

To finish the proof, we use our crafty choice of X. Write $e_t = T_t \circ e_0$, where $T_t(x,s) = (x, s+t)$ is a diffeomorphism of $D \times \mathbb{R}$ to itself. Then T_t is actually the associated flow of X. Viewing $D \times [0,1]$ as a subspace of this larger space, we have, by property of Lie derivatives, $e_t^* \mathcal{L}_X(\omega) = e_0^* T_t^* \mathcal{L}_X(\omega) = e_0^* \frac{\partial}{\partial t} T_t^* \omega = \frac{\partial}{\partial t} e_0^* T_t^* \omega = \frac{\partial}{\partial t} e_t^* \omega$.

Therefore, $(hd + dh)(\omega) = \int_0^1 \frac{\partial}{\partial t} e_t^* \omega dt = e_1^*\omega - e_0^*\omega$, so h is indeed a chain homotopy between e_1^* and e_0^*. ☐

Remark. Part of the motivation for this proof strategy is in the similarity between the definition of a chain homotopy and in Cartan's formula for the Lie derivative, in that we are "adding" two different paths from $\Omega^p(D)$ to itself. The comparisons between the fundamental theorem of calculus and the equation $e_1^* - e_0^*$ are also shown. Perhaps then this is a "fundamental theorem of homotopies"

Lemma 3. *Chain homotopic maps on cochain complexes induce the same map on cohomology*

Proof. We use $\Omega^*(C)$ and $\Omega^*(D)$ to represent our cochain complexes here for clarity, even though this lemma is more general.

If $f^*, g^* : \Omega^*(C) \to \Omega^*(D)$ are chain homotopic, then in degree p, $f^* - g^* = h_{p+1}d_{A,p} + d_{B,p-1}h_p$, where the $h_i : A_i \to B_{i-1}$ are $\mathbb{R}$ linear maps. Let ω be any closed p form $\in \Omega^p(C)$. We see that $(f^* - g^*)(\omega) = h_{p+1}(d_{A,p}\omega) + d_{B,p-1}(h_p\omega) = d_{B,p-1}(h_p\omega)$ since ω is closed and h is $\mathbb{R}$ linear (generally, it may just be $\mathbb{Z}$ linear). Then in cohomology, $(f^* - g^*)([\omega]) = [d_{B,p-1}(h_p\omega)] = [0]$. Therefore, $f^*([\omega]) = g^*([\omega])$ for any closed ω and any degree p. ☐

Lemma 4. *If $f : D \to C$ is a homotopy equivalence with homotopy inverse g, then $f^* : H^*_{dR}(C) \to H^*_{dR}(D)$ is an isomorphism of cochain complexes*

Proof. One checks that $(f \circ g)^* = g^* \circ f^*$. Then since $f \circ g \sim id_C$, $(f \circ g)^* = g^* \circ f^*$ and $id_C^* = id_{\Omega^*(C)}$ induce the same maps on cohomology by lemmas 2 and 3. Likewise, $f^* \circ g^* = id_{\Omega^*(D)}$ on cohomology. The identity maps on differential forms certainly induce the identity maps on cohomology. Therefore, f^* and g^* induce inverse isomorphisms on cohomology, and so $f^* : H^*_{dR}(C) \to H^*_{dR}(D)$ is an isomorphism of cochain complexes. ☐

Corollary 5. *(Poincaré Lemma) A star shaped domain D of $\mathbb{R}^n$ has the same De Rham cohomology as a single point. In particular, $H^0(D) = \mathbb{R}$ and all other $H^i(D) = 0$.*

Proof. Combine lemmas 1 and 4 ☐

Remark. Of course, this proof works the same if there is any deformation retract at all of D onto a point, whether or not it is as simple as the radial one described in Lemma 1. Such a domain is just called contractible, and this is a much broader set of shapes than the star shaped ones. Take your favorite star-shaped domain and start smoothly stretching/squashing it. You have created a contractible domain via homotopy equivalence.