Modular Representation Theory of Finite Groups Seminar Syllabus

Today 2025

We will explore modular representation theory of finite groups.

All talks will be recorded but will not be published unless the speaker explicitly requests it. Please contact Sam if you would like access to the full (unlisted) playlist of videos.

Feel free to talk to Sam if you have questions about one of the talks.

1. **Introduction to Modular Representation Theory**
   Highlights: Motivation for the seminar given in terms of refining classification of finite simple groups. "p-local" group and representation theory introduced. Failure of Maschke's Theorem discussed. Correct count of simple modules given, and these are stated to all be socles of projectives. Block decomposition discussed. The rest of the seminar series is outlined.
   References: [5] ch. 1,2,3, 6. This draws heavily from [14], [1]. See also [15] section 1. Date: Jan 15 2025; Speaker: Sam
   Advice to Speaker: Stay frosty

2. **The cde Triangle**
   Highlights: The Cartan matrix over finite fields is introduced and given as a motivation. The technique of lifting to characteristic zero is explained, and the decomposition matrix introduced. The cde triangle and $C = D^t D$ result are explained. This is all applied to block decompositions. Example computations are given.
   References: [15] section 3-5 for the block and character theory and examples. [5] ch 4 for nice explanation of cde. Some nice examples are also done in [16]. Even more examples done in [9]. Most of the theory goes back to [14]; Date: Jan 22 2025;
   Speaker: Luna
   Advice to Speaker: You should assume that your characteristic zero field is "large enough" for the group algebra to be split semisimple. You should avoid mentioning Brauer characters explicitly since the same ideas are already expressed with the lifting and block theory. You can avoid most proofs, but you should at least explain why the cde maps are well-defined and why d and e are adjoint. As important as the theorem is, no one will actually believe it unless you do one concrete example showing $C = D^t D$.

3. **Defect Groups & the Brauer Correspondence**
   Highlights: Induction of subgroups reviewed. Relative projectivity and vertices defined. Defect groups defined. Brauer's First Main Theorem introduced and discussed.
   References: [15] section 2 and 6 for the overview of main ideas, [1] ch 13 and 14 and [5] for proofs and more exposition. I highly recommend also watching the first part of [16]; Date: Jan 29 2025;
   Speaker: Colin
   Advice to Speaker: Attempt some examples from [5] if you have time - especially if few examples were done in the previous talk. There are a few different (equivalent) definitions of the things in this lecture, but I recommend sticking to those given in Taylor's notes for simplicity. We will not need any of the relevant technical lemmas after this talk, so only prove those that you really have time for. The main properties of the defect group and the Brauer correspondence are the most important things to try

and prove. You should really highlight the “p-local” nature of the results, i.e. how much of this is controlled by groups $N_G(P)$.

4. **Symmetric Groups**
Highlights: Non-modular representation theory of the symmetric group reviewed. Specht modules and Young Tableaux introduced. It is explained why things are different in the modular case. p-regular partitions introduced and related to the parameterization of simples in modular case.
References: [17] for char 0, [7] for char p. Characteristic zero ideas are pretty standard and can be found in many places. See [4] ch 2 for more advanced tools. Some of these refer back to a text of James and Kerber that I had trouble getting access to. Date: Feb 12 2025;
Speaker: Ian
Advice to Speaker: This talk is moreso meant to highlight explicitly how and why combinatorial things can be different in different characteristics. There are plenty of related combinatorial or algebraic results in the sources; you can do whichever you think will work best. Much more modern categorical tools are in Brundan/Kleshchev's notes and the ideas here can be used to hint at the upcoming conjectures, but don't worry about this if it is too difficult.

5. **Representations of Finite Lie-Type Groups in Defining Characteristic**
Highlights: Simple representations of connected, simply-connected algebraic groups classified. Bijective proof of Alperin's conjecture for finite simple Lie-type groups explained in these terms. References: [13] gives the overview and the explanation of the bijective proof, see also [12]. The geometric background is covered in more detail in [10] Date: Feb 19 2025;
Speaker: Tom Han
Advice to Speaker: The point of this talk is really to explain the bijective proof in as much detail as possible. You can review as much geometry or group theory as you feel is helpful. You should emphasize that it is easy to just say that the sets have the same size, and hard to explain why. You don't have to state Alperin's weight conjecture in full generality, since this will be next week. You can allude to Deligne-Lusztig theory for similar ideas in non-defining characteristic, although this is pretty complicated and best left as a bonus talk

6. **Brauer Trees**
Highlights: Decomposition matrices reviewed. Brauer trees and Brauer tree algebras defined. It is shown that Brauer trees determine the decomposition/Cartan matrices, and in fact a Morita equivalence, when the defect group is cyclic. Examples given.
References: [15] section 8 for some Morita theory background and motivating calculations. [1] part 5 for a comprehensive development of Brauer trees, although you won't need most of this. The original paper [3] is fairly short and has good motivation, although is fairly old. See [11] and [8] for other important papers. Date: Prerecorded;
Speaker: Sam
Advice to Speaker: I would consider this talk successful if you can demonstrate nicely how to obtain the decomposition matrix from the Brauer tree. It is very nontrivial to prove that the relevant blocks are even Brauer tree algebras- generally I suggest avoiding details on statements that don't have a nice combinatorial/character theoretic interpretation. You should do at least one concrete example completely i.e. both the tree and the matrices. The sources are pretty light on where the Morita equivalence comes from.

7. **Alperin and Broue's Conjectures**
Highlights: Height is introduced, and Alperin-McKay conjecture is stated. Alperin's weight conjecture is also introduced. Broue conjecture is given along with supporting calculations. Relationship among the conjectures (when defect group is cyclic) is discussed, as well as current status.
References: [15] section 7-9 for the first piece. I also highly recommend the second piece of [16]. The weight conjecture is probably best explained in [13]. Raphael probably knows the most about how the conjectures are related. Date: Feb 28 2025;

Speaker: Kiran
Advice to Speaker: Try and relate the conjectures as much as possible to the previous 3 talks and to each other. Broue's conjecture specifically needs careful motivation since there is little apparent reason to introduce the derived category. Skip Alperin-McKay if needed. You should emphasize that some of these conjectures are still very theoretically mysterious even though several cases are known.

8. **The Stable Module Category**
Highlights: Motivation of computing group cohomology given. Frobenius-ness of kG proven in speaker's favorite way. Basics of triangulated categories reviewed. The stable modular category is defined, with its triangulated structure roughly outlined. The tensor structure is also shown.
References: [6] ch 1-5. Basics on triangulated categories or Frobenius algebras can be found in many places. Date: Mar 5 2025;
Speaker: Emmy
Advice to Speaker: The central point to try and get across is why the stable category is an interesting/important thing to look at. Don't do any proofs that use actual homological algebra. A lot of this material is somewhat standard, so feel free to present things however makes sense to you. If you have time, you might mention the relationship to the usual derived category. A lot of this will be review for people who have taken 212 recently but that's okay. The finer structure of group cohomology is covered next week. You may prefer to coordinate with the next speaker if you want to go more or less in detail. You may also want to call back to some earlier computations in the seminar where there were inconvenient projectives popping up in calculations.

9. **Group Cohomology & Support Varieties**
Highlights: The product on Ext groups is introduced. The cohomology ring is reviewed and quickly shown to be graded-commutative. A few basic examples given. The Evens-Venkov theorem is stated. Support varieties of modules are introduced, and basic properties stated.
References: [6] ch 6 and 7 for cup products and examples, ch 9 and 10 for ideals and varieties in cohomology. See also [2] and references theirein for a modern tt-geometry approach. You could ask Paul if not for the fact that he is on sabbatical. Date: Mar 12 2025;
Speaker: Jon
Advice to Speaker: There are several (equivalent) definitions of the cup product. I suggest picking one of the natural composition-based ones and sticking with it. You can throw out odd-degree pieces of cohomology to make the ring commutative. You can do a lot of this talk with modern tt-geometry support theory if you prefer, but you should explain why this is related to group cohomology. I advise against most proofs for simplicity and time.

References

[1] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986, Modular representations as an introduction to the local representation theory of finite groups. MR 860771

[2] Paul Balmer, *Supports and filtrations in algebraic geometry and modular representation theory*, Amer. J. Math. **129** (2007), no. 5, 1227–1250. MR 2354319

[3] Richard Brauer, *Investigations on group characters*, Annals of Mathematics **42** (1941), no. 4, 936–958.

[4] Jonathan Brundan and Alexander Kleshchev, *https://pages.uoregon.edu/brundan/papers/durham.pdf*.

[5] Daniel Bump, *http://sporadic.stanford.edu/modrep/*, 2015.

[6] Jon F. Carlson, *Modules and group algebras*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1996, Notes by Ruedi Suter. MR 1393196

[7] Charlotte Chan, *https://web.math.princeton.edu/~charchan/ModularRepresentationsSymmetricGroupSeminar.pdf*.

[8] E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. (2) **84** (1966), 20–48. MR 200355

[9] Arun Debray, *https://adebray.github.io/writing/adebray_senior_thesis.pdf*.

[10] James E Humphreys, *Modular representations of finite groups of lie type*, vol. 326, Cambridge University Press, 2006.

[11] G. J. Janusz, *Indecomposable modules for finite groups*, Ann. of Math. (2) **89** (1969), 209–241. MR 244307

[12] Raphael Rouquier, *https://www.math.ucla.edu/~rouquier/papers/ias.pdf*.

[13] ______, *https://www.youtube.com/watch?v=8xirHL-0tZw*.

[14] Jean-Pierre Serre, *Linear representations of finite groups*, french ed., Graduate Texts in Mathematics, vol. Vol. 42, Springer-Verlag, New York-Heidelberg, 1977. MR 450380

[15] Jay Taylor, *https://www.math.ias.edu/files/broues-conjecture-I.pdf*.

[16] ______, *https://www.youtube.com/watch?v=oTPpb208pvs*.

[17] Leonard Tomczak and Stacey Law, *https://math.berkeley.edu/~ltomczak/notes/Mich2022/RepSn_Notes.pdf*.