Hall Algebras

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Inspiration

- A classic representation theory question is to investigate what properties of an object are easily visible in the category of representations
- $Rep_k(G)$ is semisimple iff $char(k)$ doesn't divide $|G|$ (Maschke)
- $Rep_{\mathbb{C}}(Q)$ is finite type iff the graph of Q is Dynkin (Gabriel)
- $Coh(X)$ for X smooth projective variety has cohomological dimension less than or equal to $dim(X)$ (Serre)

Quiver Representations

- In the Dynkin case one can obtain the positive part of the root lattice by looking at the indecomposables in $Rep_{\mathbb{C}}(Q)$
- We would like to obtain directly the algebra structure on the universal enveloping algebra $U(\mathfrak{n}_+)$
- What should "multiplication" of two quiver representations look like?

Hall Numbers

- For now, consider finite field $\mathbb{F}_q = \mathbb{F}_{p^k}$ and Dynkin Q
- $F^L_{M_1,M_2} = |\{X \subset L | X \sim M_2, L/X \sim M_1\}|$ is finite for any finite dimensional representations L, M_1, M_2.
- These can be understood as equivalence classes of short exact sequences. They are sometimes called *Hall Numbers*

The Algebra

- Define the *Hall Algebra* $H(Q, \mathbb{F}_q)$ to be the $\mathbb{C}$ algebra whose basis is the isomorphism classes of objects in $Rep_{\mathbb{F}_q}(Q)$
- The multiplication is given by $[M_1] * [M_2] = \sum_L F^L_{M_1,M_2}[L]$
- This sum is always finite since every $Ext(M_1, M_2)$ is finite dimensional
- The unit of this multiplication is $[0]$
- Fact: This multiplication is associative
- This algebra is graded by the Grothendieck group of $Rep_{\mathbb{F}_q}(Q)$

A_1

- Let $Q = * = A_1$, with one vertex and no arrows
- Representations are simply vector spaces. Finite dimensional ones are in bijection with $\mathbb{N}$.
- We compute
$[V_1] * [V_1] = P^2_{1,1}[V_2] = |P^1(\mathbb{F}_q)|[V_2] = \frac{q^2-1}{q-1}[V_2] = [2]_q[V_2]$.
- A similar computation shows $[V_n] * [V_m] = \binom{n+m}{m}_q [V_{n+m}]$
- There is an isomorphism of algebras $H(A_1, \mathbb{F}_q) \to \mathbb{C}[x]$ with $[V_n] \to x^n/[n]_q!$

A_2

- Let $Q = *_1 \to *_2 = A_2$
- There are 3 indecomposables; S_1, S_2, P_1
- We compute $Hom(S_1, S_2) = Hom(S_2, S_1) = 0$, $dim(Ext(S_1, S_2)) = 1$, $dim(Ext(S_2, S_1)) = 0$
- Therefore, $[S_2] * [S_1] = [S_2 \oplus S_1]$, but $[S_1] * [S_2] = [S_2 \oplus S_1] + [P_1]$
- Rewriting, we have $[[S_1], [S_2]] = [P_1]$

A_2 continued

- It is not so hard to check that $H(A_2, \mathbb{F}_q)$ is isomorphic to the Heisenberg algebra
- We will have

$$[S_1] \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[S_2] \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[P_1] \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Possibly with some rescaling

Relation to Lie Theory

- The underlying graph of A_2 describes the root system for $\mathfrak{sl}_3$
- Note that the Heisenberg algebra is $U(\mathfrak{sl}_{3,+})$
- Likewise, $\mathbb{C}[x] = U(\mathfrak{sl}_{2,+})$
- Proving this phenomenon generally is the goal of Ringel's Theorem

Key Observations

- Amazing Fact: $F^L_{M_1,M_2}$ only depends on the associated map $R_+ \to \mathbb{Z}_+$ for L, M_1, and M_2
- Amazing Fact: $F^L_{M_1,M_2}$ is actually an integer polynomial in $q = |\mathbb{F}_q|$
- This allows us to define a universal Hall algebra $H(Q)_A$, where $[M] * [N] = \sum_l F^{f_L}_{f_M,f_N}(t)[f_L]$
- We specialize $H(Q)_A$ by mapping t to 1. We call this algebra $H(Q)$.

Generators

- We were able to show in the previous examples that the simple representations actually generate $H(Q)$. We will want to show that this is true generally
- Due to AR theory, there is a total order on the indecomposable representations such that $Hom(A, B) = Ext(B, A) = 0$ if $B < A$. This is basically a refinement of the "obvious" order on the AR quiver
- We can then compute that $[\bigoplus_{k=1}^{l} n_k I_k] = \prod_{k=1}^{l} [I_k]^{(n_k)}$, assuming that the indecomposables are ordered

More on Generators

- An induction argument using this order also lets us prove that all indecomposables are in the subalgebra generated by the $[S_i]$
- Therefore, $H(Q)$ is generated by the $[S_i]$
- Note that this is NOT true if Q is not Dynkin. Generally, we define $C(Q) \subset H(Q)$ the composition algebra to be the subalgebra generated by the $[S_i]$.

The Homomorphism

- $U(\mathfrak{n}_+)$ is also generated by simple elements e_i. We would like to define an algebra homomorphism sending e_i to $[S_i]$
- This amounts to checking the Serre relations for the $[S_i]$

$$[e_i, e_j] = 0 \text{ if } i \text{ and } j \text{ are not connected}$$
$$[e_i, [e_i, e_j]] = 0 \text{ if } i \text{ and } j \text{ are connected}$$

- Surjectivity of this map follows from the prior slide
- Injectivity follows from a graded dimension computation via PBW theorem

Hopf Algebra Structure

- $U(\mathfrak{g})$ has the structure of a Hopf algebra. How should we interpret the coproduct and scalar product?
- The coproduct will have the form $\Delta([R]) = \sum_{M,N} c_{M,N}[M] \otimes [N]$
- This may require passing to a formal completion of the Hall algebra to deal with infinite sums in certain categories
- Counit $\epsilon([M]) = \delta_{M,0}$
- $([M],[N]) = \delta_{M,N}/|Aut(M)|$
- Antipode also exists - this can be upgraded to a genuine (topological) Hopf algebra structure

Other Categories

- Definition of Hall algebra does not explicitly use quivers - just some finiteness conditions
- We generally require Hom-finiteness: $Hom(M, N)$ and all $Ext^i(M, N)$ are finite. Easily achieved if we look at finite fields.
- Generally need finite global cohomological dimension. The most tractable cases are the hereditary categories i.e. $Ext^k = 0$ for $k \geq 2$
- This will allow us to construct Hall algebras for non-Dynkin quivers and coherent sheaves on curves

Non-Dynkin Quivers

- We often impose that representations of Affine quivers be nilpotent so that we have good finiteness
- In this case, the map $U(\mathfrak{n}^+) \to H(Q)$ is still well-defined and injective, but will not be surjective in general. The image is called the *composition algebra*
- Recall that indecomposables were called either preprojective, preinjective, or regular in this case
- One can show that $H(Q) = H_P \otimes H_R \otimes H_I$ and $C(Q) = C_P \otimes C_R \otimes C_I$

Coherent Sheaves

- We mostly focus on finite fields again
- $Coh(\mathbb{P}_1)$ is also hereditary. Generally, $Coh(X)$ for X a smooth curve
- If $g(X) \geq 2$, then this category is wild! Only $\mathbb{P}^1$ is really understood well, and elliptic curves are still being explored
- Lenzing introduced certain one dimensional subvarieties of weighted projective spaces which are not smooth but are homologically nice

$Coh(\mathbb{P}^1)$

- The indecomposables here are the line bundles and the simple torsion sheaves. Recall $Tor(\mathbb{P}^1)$ is abelian and extension closed, but $Vec(\mathbb{P}^1)$ is not
- Kapranov proved that there is a similar isomorphism $U(\mathcal{L}b_+) \to \overline{H}(\mathbb{P}^1)'$
- This is reminiscent of the derived equivalence $Coh(\mathbb{P}^1) \to Rep(Kron)$

More Extensions and Applications

- A similar analysis can be done for the weighted projective lines, although one should use the HN filtration to understand the indecomposables
- Lusztig produces canonical bases of such algebras by categorifying
- There are many attempts to recover the full enveloping algebra, not just the positive part
- One can perform a Hall algebra construction on certain cluster categories and recover cluster algebras.

- Kirillov "Quiver Representations and Quiver Varieties"
- Schiffmann "Lectures on Hall Algebras"