The Universal Bundle: Staying Organized for the GT Exam

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Contents

Intro

Many of the incoming math graduate students here at UCLA end up taking the geometry/topology qualifying exam during their first or second year. Not many end up researching geometry or topology, but the topics are an accessible middleground between algebra and analysis, and the exam itself has a reputation for being easy if you are prepared.

However, despite this, I have heard plenty of complaints about certain topics or techniques that appear regularly on the exam but are not covered well (or at all) in the preparatory course sequence. The expectation in preparing for this exam is that students will review the (publicly available) past exams to see what kinds of questions can occur, and study accordingly. Unfortunately, this is sometimes *all* a student can do because the 225 sequence is never totally comprehensive. I don't like having to recommend to fellow students that they should "just memorize every previous problem" in order to be be best prepared. The point of this guide here is to bring awareness to and keep organized some of the more uncommon topics.

To be clear, this guide is **NOT** intended to replace anything in the 225 sequence, or even to replace one's individual studying of these very topics. This document is designed as an organizational tool so that one's studying can be more efficient than "memorize every previous problem". The goal here is that you might actually *learn* some of these more esoteric topics. In order to get the most out of this guide, you will need to do your own due diligence in studying and learning the relevant material, using other sources when needed.

The standard readings for the exam are Lee's *Introduction to Smooth Manifolds*, Guillemin and Pollack's *Differential Topology*, and Hatcher's *Algebraic Topology*, each receiving about one quarter each in the standard 225 sequence. If you are comfortable with the material from these texts as it appears on the exam, then I would say you generally do not need to look for other sources of knowledge. Professors Ko Honda and Peter Petersen both have lecture notes about manifold theory on their websites in case you would like extra sources for these materials. In my opinion, they don't do anything new or different from Lee, but are much more compact, so this may help you. I don't think there are many more accessible texts for algebraic topology over Hatcher, but there are plenty that are more algebraic (Spanier, tom Dieck, and my favorite May). You might have to ask other students for more recommendations.

I have some personal recommendations about these three main sources. Lee's text chapters 1-19 are all used in some context, although most of the content of these chapters is very techincal proofs. Lee enjoys defining all of his atlases, topologies, and extensions very explicitly, but this level of detail is not required for the exam. Only a few of the problems here are especially helpful, but the calculations done in examples are usually enlightening. I am really not a fan of the extremely brief style of exposition in Guillemin and Pollack. Certain chapters actually leave all of the proof up to you. The problems here vary from pedantic detail checking to problems that have appeared explicitly on the exam. My personal recommendation is to just figure out which problems these are and not spend too much time here. Hatcher chapters 0-3 + certain appendices are all covered, with an emphasis on the main topics in chapters 0-2. This book is worth reading carefully, since many examples and problems from here have appeared on the exam.

That said, text sources only get you so far. On average, the most time consuming part of any student's preparation for the exam is in reviewing old problems and their solutions. **You are generally ready to begin this phase once you are comfortable with the definitions and statements of main theorems**. Qualifying exam problems are often pretty different from the kinds of things that you see in the 225 homeworks or in the texts, so don't spend too much time elsewhere.

The content of this guide is a brief review of the key definitions, theory, and techniques behind several minor exam topics. My coverage is far from exhaustive, and I generally recommend doing more studying beyond what you see here. Study until it makes sense *to you*. I also gather many previous exam problems on each topic at the end of each section, along with problems from other sources. I do not provide solutions to most of these, since solution guides are generally available elsewhere.

This guide is written assuming that you have familiarity with the content of the 225 sequence; it would be hard to explain symplectic manifolds if you did not already know what a manifold is. Each section of the guide begins with a box of the following sort.

Suggested Reading

In a box like this, I will suggest a few possible alternative reading materials, as well as my opinions of them. "Suggested" may be a stronger word than I really mean; these readings sometimes just repeat the information I provide or are too terse to read carefully. But I also appreciate that everyone comes into this exam with a different background, so I do what I can to give you options.

The biggest feature of each section will be proofs of some of the theorems and techniques that appear on the exam.

This is something you should prove

Any fact or theorem that is usually worth *proving yourself on the exam* will appear in a red box like this. You will have to use your own discretion as to whether the particular problem you are given expects you to prove the red-box result or to just state it. Some of these problems are intermediate lemmas or results that are not generally taught in the courses or textbooks, so are not "given" knowledge. Others are problems that have actually appeared on the exam before that other problems may be based on. After having read through my version of the proof, it may be worth trying to find a proof that you are comfortable with and would be willing to write down on the exam.

Periodically and at the end of a section, I give some practice problems so that you can apply the theory you have learned. These problems will come in 3 different types.

Exercise

Problems in green boxes are usually quick verifications. They are not really crucial to your knowledge, and are often just details that I don't feel like writing myself. That said, the results in them are generally worth knowing.

Old qual problem

Problems in blue boxes have actually appeared on the exam before. I will list the exams and problem numbers that I can find. Some of the red-box problems are also old exam problems and are numbered similarly. I do not provide solutions to blue box problems, and I recommend that you only look for solutions after attempting them yourself. Some problems require prior knowledge beyond what I show in each section, so don't be frustrated if you can't solve one.

Bonus

Problems in yellow boxes are interesting problems on the material that have not (yet) appeared on the exam. They are often from textbooks or are just generally known results. Some are of the general style of exam problems, and others are just there to expand your perspective on the subject matter. I will try to list a specific source for these when I can.

If you have any feedback at all on this guide, including how I can improve explanations or add extra topics, feel free to let me know! The point of this document is that it should actually be useful to people. I am trying to stay away from writing up topics that are usually covered well in the 225 sequence or in the standard texts (Frobenius theorem, transversality, singular cohomology), but would certainly look into it if asked.

As always, preparing for qualifying exams is a community effort. I could not have passed my own exams without the advice and support of my fellow students. I would specifically like to thank Matt Gherman, Eilon Reisin-Tzur, Ben Spitz, Jerry Luo, and Yan Tao for everything they have contributed to our knowledge and preparation.

Symplectic Forms

Suggested Reading

Most notes on symplectic geometry go well beyond what is needed for the exam, so further reading is overkill. That said, Lee's *Introduction to Smooth Manifolds* chapter 22 or McDuff and Salamon's *Introduction to Symplectic Topology* chapters 2 and 3 may give you more intuition or examples.

Symplectic geometry is a very active area of mathematics that combines geometry, differential equations, and even some quasi-categories. You don't need to know any proper symplectic geometry for the GT exam. Symplectic geometry happens to be one of Ko Honda's areas of research, hence why these questions appear so frequently.

Definition. A **symplectic form** on a $2n$-dimensional smooth manifold M is a closed 2-differential form ω on M such that ω^n is nowhere zero. A manifold M admitting a symplectic form is called a **symplectic manifold**, and is denoted (M, ω).

There exist other definitions of symplectic forms that are more useful for answering geometric questions, but this is the only definition that seems to appear on the exam. The following two examples are the "canonical" examples that most other symplectic forms on the exam tend to resemble.

Example. Let $M = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, x_2, y_2, ...x_n, y_n)$. Then $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic form on $\mathbb{R}^{2n}$. You should verify that this is indeed closed and that $\omega^n = n!dV = n!(dx_1 \wedge dy_1 \wedge dx_2 ... \wedge dy_n)$. Notice that this ω can be written as a sum of forms $\omega_1 + ...\omega_n$, where ω_i is a symplectic form on $\mathbb{R}^2_i$.

Example. Let $M = S^2$. Then any volume form (defined as a nowhere vanishing top degree form) is symplectic.

Actually, a consequence of Darboux's theorem is that *every* symplectic manifold is locally symplectomorphic to the one given in the first example, although you don't really need to know this. The point to keep in mind is that on a manifold that looks like a product, you can try adding symplectic forms from each coordinate, which may often be volume forms on their respective component.
While this definition seems fairly basic, it turns out that **most manifolds do not admit symplectic forms**. Many exam questions are based on this idea; given the existence of a symplectic form, the number of manifolds that you could really be looking at is greatly reduced. There are a few key principles that influence whether a certain manifold admits a symplectic form.

Fact. Symplectic manifolds are always even dimensional.

This is baked into the given definition, but not into some other definitions. Keeping this in mind sometimes helps you cross out certain possibilities.

Fact

Symplectic manifolds are orientable.

This is because a manifold is orientable exactly when it admits a nonvanishing top-degree form. Due to this, Poincare duality applies, and so the deRham cohomology of a symplectic manifold can be studied with greater precision. Note that, **because ω is closed, it has a well-defined cohomology class** $[\omega] \in H^2(M)$. In fact, since we can compute derivatives of wedge products, **every power of ω is closed, and therefore has a well-defined cohomology class** $[\omega^k] \in H^{2k}(M)$. This leads to the most useful technique across all of these problems.

Lemma

If (M, ω) is a closed (compact + no boundary) $2n$-manifold, then $H^{2k}(M) \neq 0$ for any $k \leq n$.

Proof. Since M is closed, and is orientable by remarks above, Poincare duality gives us an isomorphism of real vector spaces $H^{2n}(M) \to \mathbb{R}$, $[\tau] \to \int_M \tau$. So, since ω^n is nowhere zero, $\int_M \omega^n \neq 0$, which means that $[\omega^n] \neq 0 \in H^{2n}(M)$. The wedge product descends to a well-defined linear map on cohomology. So, for any $1 \leq k \leq n$, note that $[\omega^k] \wedge [\omega^{n-k}] = [\omega^n] \neq 0$. Since this wedge product is linear, we must have $[\omega^k] \neq 0 \in H^{2k}(M)$. Of course, $H^0(M) \neq 0$ anyways. $\square$

If you understand this technique, then you are essentially prepared for every symplectic problem that will appear on the exam. It may help to keep in mind a few techniques for computing deRham cohomology, such as Mayer-Vietoris, Kunneth formula, universal coefficients, etc.

SP 18 # 5

A symplectic form on an eight dimensional manifold is a closed 2-form ω such that ω^4 is a volume form. Determine which of the following admits a symplectic form: S^8, $S^2 \times S^6$, $S^2 \times S^2 \times S^2 \times S^2$.

SP 20 # 2

Let M be a 4-dimensional manifold. A sympletic form is a closed 2-form ω such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.

1. Construct a symplectic form on $\mathbb{R}^4$
2. Show that there are no symplectic forms on the unit sphere S^4.

SP 22 # 1

Let M be a closed (= compact without boundary) $2n$-dimensional manifold and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, the map $T_pM \to T_p^*M$, $X \to i_X\omega(p)$ is an isomorphism. Show that the de Rham cohomology groups $H^{2k}_{dR} \neq 0$ for $0 \leq k \leq n$.

FA 22 # 5

Let M be a $2n$-dimensional manifold. A symplectic form on M is a smooth closed 2-form $\in \Omega^2(M)$ so that $\omega \wedge \omega \wedge ... \wedge \omega \in \Omega^{2n}(M)$ is a volume form (that is, nowhere vanishing). Determine all pairs of positive integers (k, l) so that $S^k \times S^l$ has a symplectic form.

Extra

Which torii $(S^1)^n$ are symplectic manifolds? Which products $(S^2)^n$ are symplectic manifolds?

Extra (from http://staff.ustc.edu.cn/ wangzuoq/Courses/15S-Symp/Notes/Lec02.pdf)

Let (M, ω) be a symplectic manifold. We define a **symplectic submanifold** to be an embedded submanifold $N \subset M$ for which $\omega|_N$ is a symplectic form on N.
Show that every symplectic submanifold of $(\mathbb{R}^{2n}, \omega)$ is not compact.

Extra

Let X be any smooth manifold. Show that the cotangent bundle T^*X is a symplectic manifold.
Hint: You can probably guess what the form should be, but showing it is actually well-defined is why you really need to know how the cotangent bundle coordinates work. You can check the above texts if you get stuck here, this is more just an extra practice/example.

I am of the opinion that nearly all of the basic questions that one could ask about symplectic forms and manifolds have been exhausted. I would not be too surprised if some wholly original topic related to symplectic geometry appears on the exam in the future.

Euler Characteristic

Suggested Reading

There isn't any one good source on the Euler characteristic identities. May's *A Concise Course in Algebraic Topology* chapters 4 and 21 may contain the most of them. The wikipedia page https://en.wikipedia.org/wiki/Euler_characteristic is somewhat comprehensive, although essentially just lists results.

The Euler characteristic $\chi(X)$ is one of the oldest and most broadly studied topological invariants. There isn't any good conceptual meaning for this invariant. It is perhaps best understood as a greatly condensed version of the topological data of any space. One of the reasons that this invariant is so useful is that **there are many different ways to compute it**. Only a few of these are worth proving on an exam. Many exam problems can be approached with a proof by contradiction; if some such space exists, then you can compute the Euler characteristic in two different ways and obtain different answers.

Problems involving the Euler characteristic can be tricky since they often require a mix of algebraic and differential techniques. Moreover, some problems will not even specify that the Euler characteristic should be part of your proof. You may only be asked to show that some vector field/boundary does not exist, and this can often be done by showing that the Euler characteristic would be an algebraic obstruction.

At the end of the day, these problems boil down to memorizing the identities and recognizing when an all-powerful algebraic invariant will be useful. My general philosophy here is that **"Any problem that asks you to show that something *doesn't* exist is an algebra problem"**, meaning that invariants like $\chi(X)$ are especially useful for deriving contradictions. This is true even for classes of problems that don't use the Euler characteristic.

Definition (classical)**.** For a polyhedral surface P, we have that the **Euler characteristic** $\chi(P) = V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces.

You will almost certainly not need the classical formula for the exam, although the following thought exercise may help you get into an invariant mindset.

Exercise

Compute χ for each of the 5 regular polyhedra. Come up with a quick argument for why they ought to all be equal.

Definition (standard)**.** For an n-dimensional simplicial (CW) complex X, $\chi(X)$ is the alternating sum $\sum_{i=0}^{n}(-1)^i b_i$, where b_i is i-th Betti number, which is the number of i-simplices (cells).

This is the definition to keep in mind for the exam, since you can compute it quickly for many spaces (spheres, torii, projective spaces, etc.) Some homological algebra magic gives us the next definition, which makes doing proofs much easier. See Hatcher page 146 for a proof.

Definition (homological)**.** For an n-dimensional simplicial (CW) complex X, $\chi(X)$ is the alternating sum $\sum_{i=0}^{n}(-1)^i \mathrm{rank}(H_i(X,\mathbb{Z}))$, where the rank is the number of $\mathbb{Z}$ summands in $H_i(X,\mathbb{Z})$. So, for example, $\mathrm{rank}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) = 1$.

We observe that the homological formula allows us to compute χ directly from knowledge of the homology groups. These groups are homotopy invariant, and therefore,

Fact. $\chi(X)$ is homotopy invariant.

While the Euler characteristic does exist for any complex, it is computed most often for manifolds, in part due to a larger bag of available tricks and ways to decompose these spaces. For starters, we often prefer to work with real coefficients when dealing with manifolds, which makes the following propositions useful.

Exercise

If F is any field (usually $\mathbb{R}$ or $\mathbb{Z}/2\mathbb{Z}$), show that $\chi(X) = \sum_{i=0}^{n}(-1)^i \text{rank}(H_i(X;F))$.

Lemma

If F is any field (usually $\mathbb{R}$ or $\mathbb{Z}/2\mathbb{Z}$), show that $H_i(X;F) = H^i(X;F)$. In particular, taking $F = \mathbb{R}$ gives the De Rham cohomology for a manifold.

Proof. The universal coefficients theorem gives that $H^k(X;F) = Hom_F(H_k(X;F),F) \oplus Ext_F(H_{k-1}(X;F),F)$. Since F is a field, all vector spaces over F are projective, and hence the Ext group vanishes. Moreover, the F-dual of $H_k(X;F)$ is isomorphic to $H_k(X;F)$. So, we have $H^k(X;F) \sim H_k(X;F)$, as desired. □

The universal coefficients theorem is generally good to keep in mind for these problems. Beyond this, there are two particular tricks that comes up often in proofs on the exam. The first makes computing this Euler characteristic much easier for compact manifolds.

Lemma

If M is a compact F-oriented n-manifold without boundary and F is a field, then $H_k(M;F) = H_{n-k}(M;F)$.

Here, being either $\mathbb{Z}$ or $\mathbb{R}$-oriented is the same as being oriented in the usual sense. Every manifold is $\mathbb{Z}/2\mathbb{Z}$-oriented. So, you can apply the $\mathbb{Z}/2\mathbb{Z}$ version of this result even for nonorientable manifolds.

Proof. Since M is compact, Poincare duality gives that $H_k(M;F) = H^{n-k}(M;F)$. By the above lemma, this is $H_{n-k}(M;F)$. □

Corollary

If M is an odd dimensional compact manifold without boundary, then $\chi(M) = 0$.

Proof. We choose a field F for which M is orientable (for example, $\mathbb{Z}/2\mathbb{Z}$ always works). The above result gives that $\chi(M) = \sum_{i=0}^{n}(-1)^i \text{rank}(H_i(M;F)) = \sum_{i=0}^{(n-1)/2}(-1)^i \text{rank}(H_i(M;F)) + \sum_{i=(n+1)/2}^{n}(-1)^i \text{rank}(H_{n-i}(M;F)) = \sum_{i=0}^{(n-1)/2}(-1)^i \text{rank}(H_i(M;F)) + \sum_{i=0}^{(n-1)/2}(-1)^{n-i} \text{rank}(H_i(M;F)) = 0$ □

The second trick involves a somewhat nonintuitive construction that you would likely not figure out for yourself. Lee's textbook gives an overview of how this construction is actually possible in general, if you would like a reference.

Lemma, also SP 16 # 4, also SP 22 # 9

Let M be an odd dimensional compact manifold with boundary ∂M. Then $\chi(\partial M) = 2\chi(M)$.

Proof. Let N be the double of M. This is the manifold obtained by taking two copies of M, say M_1 and M_2, and gluing them along the boundary. (For a specific example, if we glue two copies of $[0,1]$ together along their boundary, we obtain the circle S^1. If we glue two copies of the closed disk D^2 along their boundary, we obtain the sphere S^2.) N is compact and odd dimensional since M is. By the above result, $\chi(N) = 0$. We choose a decomposition of N with $A = M_1 + \epsilon$ and $B = M_2 + \epsilon$. Then A, B both deformation retract onto M, $A \cup B = N$ and $A \cap B$ deformation retracts onto the points of gluing, which is ∂M. The Mayer-Vietoris sequence for this decomposition reads $0 \to H_n(\partial M;F) \to H_n(M;F) \oplus H_n(M;F) \to H_n(N) \to H_{n-1}(\partial M;F) \to ...H_0(M;F) \oplus H_0(M;F) \to H_0(N;F) \to 0$. The alternating sum of dimensions in any long exact sequence of vector spaces is zero (you could see this by iteratively performing quotients on the leftmost spaces). We compute that this equation is $0 = \chi(N) - 2\chi(M) + \chi(\partial M)$. Since $\chi(N) = 0$, we have $\chi(\partial M) = 2\chi(M)$. $\square$

The tools above are the core tricks that you would use to compute Euler characteristics. Some of the more standard formulas are listed below. You could try and compute these, but some are rather laborious. Most are probably fine to just use without proof unless asked.

Fact. $\chi(X \times Y) = \chi(X) * \chi(Y)$.

Fact. If $\overline{X} \to X$ is a k-sheeted covering, then $\chi(\overline{X}) = k * \chi(X)$.

Fact. If X is a closed, orientable genus g surface, then $\chi(X) = 2 - 2g$.

Theorem. (Poincare-Hopf) If M is a compact, orientable manifold without boundary, and if V is a smooth vector field on M with finitely many zeros, then $\sum_{V(x)=0} ind(x) = \chi(M)$.

Corollary. If M is a compact orientable manifold without boundary, and if M admits a nonvanishing vector field, then $\chi(M) = 0$.

Exercise

Deduce the hairy ball theorem from Poincare-Hopf.

Theorem. (Gauss-Bonnet) If M is a compact, two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and k_g the geodesic curvature of ∂M. Then $\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$.

You will almost certainly not need the previous theorem, but it is fairly significant in geometry, and stranger things have appeared on the exam.

As alluded to at the beginning of this section, you will need a fairly broad knowledge base to approach some of these questions. On the geometric side, you will often be asked to construct specific vector fields on manifolds that may or may not vanish. On the algebraic side, you may need to use more advanced duality theorems alongside those proven above, since these generally allow for more sophisticated computations. You may not be able to solve the problems below with just the information in this section, although they are worth attempting.

FA 12 # 8

Show that there is no compact 3-manifold M whose boundary is $\mathbb{R}P^2$.

FA 15 # 1, also FA 20 # 10

Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices.

1. Show that $SL_n(\mathbb{R})$ is a smooth submanifold of $M_n(\mathbb{R})$.
2. Show that $SL_n(\mathbb{R})$ has trivial Euler characteristic.

SP 17 # 3

Use the Poincare-Hopf index theorem to calcluate the Euler characteristic of S^n (You must compute the indices in local coordinates. Drawings do not suffice!)

FA 16 # 6, also FA 17 # 7

Suppose M is a smooth, compact, connected, and oriented manifold (without boundary)

1. Show that if the Euler characteristic of M is zero, then M admits a nowhere vanishing vector field
2. If M is a surface of genus g, then what is the min_v(number of zeros of v), where v ranges over vector fields whose zeros are isolated and have index ± 1?

SP 19 # 10

Suppose M^n is a compact, connected, orientable manifold with boundary a rational homology sphere, ie. $H_*(\partial M, \mathbb{Q}) = H_*(S^{n-1}; \mathbb{Q})$.

1. Assuming n is odd, use Poincare duality to show that M has Euler characteristic $\chi(M) = 1$.
2. Assuming $n \sim 2 \bmod 4$, show that $\chi(M)$ is odd.

FA 19 # 3

For which n does the real projective space $\mathbb{R}P^n$ admit a nowhere vanishing vector field? If one exists, give an explicit one.

SP 23 # 6a

If X is a finite CW complex and $\overline{X} \to X$ is a path-connected n-fold covering map, then show that the Euler characteristics are related by the formula $\chi(\overline{X}) = n\chi(X)$.

Extra (Hatcher problem 1.A.3)

For a finite graph X define the Euler characteristic $\chi(X)$ to be the number of vertices minus the number of edges. Show that $\chi(X) = 1$ if X is a tree, and that the rank (number of elements in a basis) of $\pi_1(X)$ is $1 - \chi(X)$ if X is connected.

Extra (related to SP 18 # 7)

If X and Y are smooth, connected, orientable manifolds of dimension at least 3, compute the Euler characteristic of the connect sum $X\#Y$ in terms of that of X and Y.

Suspensions

Suggested Reading

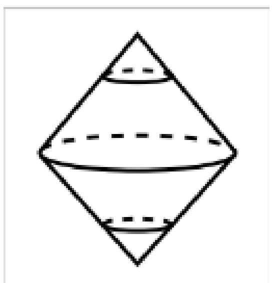
If you want to know the very basics on categories and functors, you could try Hatcher's page 162. If you *really* wanted to understand categories and functors, I recommend Riehl's *Category Theory in Context*. Most of the difficulty with suspensions is about how to relate the algebraic information to the topological constructions. Otherwise, May's *A Concise Course in Algebraic Topology* chapters 6-9 are good references for the topological material.

The suspension of a topological space X is a new topological space SX whose homology very closely resembles that of X. The algebra behind suspensions and degree raising maps is extremely important in modern algebra. Stable homotopy theory comes from computing homotopy groups of a suspension spectrum, and derived categories are defined with an operator that resembles the suspension of a chain complex. As such, algebraically-minded instructors love to put questions using suspensions on the exam. Hatcher mentions the suspension in a few places, but it is not generally emphasized.

What makes this topic so difficult is that it combines some of the most confusing pieces of algebraic and point-set topology. Analytically, a suspension is defined as a seemingly ad-hoc quotient space without any clear motivation. Algebraically, nobody likes computing relative homology, and not everyone who takes the GT exam is comfortable with the language of functors. In this section, I will try to cover the topic of suspensions in such a way that addresses both of these challenges. Hopefully, understanding this section will help you understand similar constructions, like mapping cones.

Definition. For a topological space X, the **suspension** SX is the quotient of the cylinder $X \times I$ by collapsing each end to a point. Formally, $SX = (X \times I)/((x, 0) \sim (y, 0), (x, 1) \sim (y, 1))$.

The reason this is called the suspension is because it resembles the space X being suspended between the north and south pole. See the picture below of $S(S^1)$, taken from Hatcher.



Courtesy of Hatcher

It is sometimes useful to consider the intermediate construction of the cone of X, say CX, which is $X \times I/X \times \{0\}$, where we only collapse the north pole of the cylinder. Then, we can compute $SX = CS/(X \times \{1\})$. All we have really done here is collapse the poles one at a time instead of simultaneously. Sometimes, this step is useful, since the cone is contractible. Furthermore, the suspension SX is actually the union of two cones, since the subset of SX given by $X \times [0, .5]/\sim$ is actually homeomorphic to CX. This is marked in the picture above.

The reason that the suspension is introduced is because of its homology.

Lemma, also SP 14 # 10, also FA 20 # 6, also Hatcher problem 2.1.20

For a CW complex X, $\tilde{H}_n(X) \sim \tilde{H}_{n+1}(SX)$. Furthermore, $\tilde{H}_0(SX) = 0$.

Proof. We note that SX is always path-connected since any $(x, i)/\sim$ is connected to the north pole via the path $(x, t)/\sim$. Therefore, $\tilde{H}_0(SX) = 0$. For the higher groups, we apply Mayer-Vietoris. Our open cover will be $A = X \times [0, .55)/\sim$ and $B = X \times (.45, 1]/\sim$. Both A and B are contractible, since we can push all points towards one of the poles. $A \cap B$ deformation retracts onto X itself. The Mayer-Vietoris sequence for reduced homology reads

$$...\tilde{H}_{n+1}(A \cap B) \to \tilde{H}_{n+1}(A) \oplus \tilde{H}_{n+1}(B) \to \tilde{H}_{n+1}(SX) \to \tilde{H}_n(A \cap B) \to \tilde{H}_n(A) \oplus \tilde{H}_n(B)....$$

Using the above deformation retracts, this becomes

$$...\tilde{H}_{n+1}(X) \to 0 \to \tilde{H}_{n+1}(SX) \to \tilde{H}_n(X) \to 0....$$

Exactness forces that the connecting map $\tilde{H}_{n+1}(SX) \to \tilde{H}_n(X)$ is an isomorphism; the zero on the left forces injectivity and the zero on the right forces surjectivity. $\square$

You could make a similar argument by computing the relative homology of the pair $(CX, X \times \{1\})$, since $CX/X \times \{1\} \sim SX$. Recall, as usual, that for $n \geq 1$, we have that $\tilde{H}_n(SX) = H_n(SX)$.

The underlying theme of this construction, as well as other related quotient spaces, is that homology groups are hard. To compute them, we like using long exact sequences to relate the homology of a new space to the homology groups of spaces that we already know. Spaces that deformation retract onto known spaces (especially onto a point) are usually easy to locate, and this makes our computation easy. If you want to start thinking like a homotopy theorist, then you should start reminding yourself that **contractible spaces "may as well be a point" for homotopy purposes**. There are only two long exact sequences for homology you could sanely write down; Mayer-Vietoris and the long exact sequence for relative homology. One of the two is usually sufficient for a given problem.

Once this proof has really set in, try your hand at a few of these problems.

FA 13 # 8

Let $n > 0$ be an integer and let A be an abelian group with a finite presentation. Show there is a space X with $H_n(X) = A$.

SP 16 # 9

Let X be a topological space and $p \in X$. The *reduced suspension* ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and ΣX.

The reduced suspension is sometimes referred to as the based suspension or pointed suspension. The key difference in practice is that, if X is a pointed topological space, the base point x_0 becomes stretched across the interval $[0,1]$ within the suspension SX. However, if we collapse $x_0 \times [0,1]$ down to a point, then there is a natural choice of basepoint within ΣX.

Exercise

Show that ΣX from the previous problem is homeomorphic to the smash product $X \wedge S^1$.

Bonus

Let X and Y be topological spaces, and $f : X \to Y$ be a continuous map between them. Define cone(f) to be the *mapping cone* of f, given by the set $X \times [0,1] \bigsqcup Y/\sim$, where $(x_1, 0) \sim (x_2, 0)$ for all $x_i \in X$ and $(x, 1) \sim f(x) \in Y$ for all $x \in X$. Write out a long exact sequence describing the homology groups of cone(f) in terms of those of X and Y. (Your answer will also be in terms of the map induced by f on homology)

One of the other algebraically useful facets of the suspension is that **we can suspend continuous functions**. What I mean is this; suppose we have a continuous function $f : X \to Y$. How could we define a new function between SX and SY from f? Well, we might hope that it could be defined slice-wise. So, the equatorial slice $X \times \{.5\}$ would hopefully be mapped to $Y \times \{.5\}$, since these slices are homeomorphic to X and Y respectively. Formally, given the maps $X \to^f Y$ and $[0,1] \to^{id} [0,1]$, we can define a continuous function on the product spaces $X \times [0,1] \to Y \times [0,1]$, where $(x,t) \to (f(x), id(t)) = (f(x), t)$. If we compose with the projection $Y \times [0,1] \to^p SY$, we can see that $p \circ f(x_1, 0) = p \circ f(x_2, 0)$ and $p \circ f(x_1, 1) = p \circ f(x_2, 1)$, i.e., the north slice is mapped to the north pole. The universal property of the quotient gives us a well-defined continuous map $SX \to^{Sf} SY$ where $(x,t)/\sim \to (f(x), t)/\sim$.

So, we can suspend both topological spaces and the continuous functions between them. This data, along with the following exercise, makes up the data of a **functor** from the category of topological spaces to itself.

Exercise

Check that the suspension of the identity function $id : X \to X$ is the identity of SX. Check that for two continuous functions $X \to^f Y \to^g Z$, we have that $S(g \circ f) = S(g) \circ S(f)$.

Exercise

Check that the suspension is also a functor from the *homotopy category* of topological spaces to itself. Meaning that if $f \sim g$ as continuous functions, then $Sf \sim Sg$.

You have already seen many functors in algebraic topology. The fundamental group (in fact, any homotopy group) is a functor from the category of based topological spaces to the category of groups. Homology and cohomology groups are functors (cohomology is contravariant) from the category of topological spaces to the category of abelian groups (or to some other R-mod depending on your choice of coefficients). From differential geometry, the assignment of any smooth manifold M to its tangent bundle TM is a functor from the category of smooth manifolds to itself. The smooth function $f : M \to N$ is taken to $Df : TM \to TN$. The key property of functors is that they also take functions to functions, and they preserve the composition of functions via the above exercise. We can convert data of topological spaces into data of some other kind of mathematical object, which gives us more tools to answer topological questions. Essentially, you should think of functors as being like "structure preserving functions between categories". This is not a totally incorrect way of thinking; the category of (small) categories has functors as its morphisms.

We can of course compose functors like we would compose functions. We write $S^2(X) = S(S(X))$. We can also take homology of a suspension $H_k(S(X))$. Part of what makes suspension so important is that it is a surprisingly common functor with the same source and target category, meaning it can be composed with itself as much as we want. We can keep suspending and suspending forever! As silly as it sounds, this happens to lead to stable homotopy theory due to how little information is lost with this suspending process.

Fact. $S(S^n) = S^{n+1}$

I apologize for the confusing notation, but this is exactly what Hatcher uses. The leftmost S is a suspension, and the others mean spheres. These are the only concrete suspensions that are really worth knowing.

If you were to take anything away from this discussion at all, it is that **suspension raises the degree of homology**. One could write an entire note sheet about the ins and outs of suspensions and a build up to stable homotopy theory, but you will only need the basics for the exam. I won't keep you in suspense any longer, here are the rest of the problems.

FA 22 # 10

Let $f : X \to Y$ be a continuous, pointed map. Let $\Sigma^n(f) : \Sigma^n X \to \Sigma^n Y$ be the nth (pointed) suspension of f. Show that if for some n, $\Sigma^n(f)$ induces the trivial map on reduced homology, then it does for all n.

Bonus

Describe a natural (and nontrivial) function $\pi_k(X) \to \pi_{k+1}(\Sigma X)$. Can you show that it is a group homomorphism?

Bonus

Define the *loop space* ΩX of a pointed topological space (X, x) to be the space of pointed continuous maps $Map_*((S^1, 1), (X, x))$ with the compact-open topology. This space is pointed with base the constant function $f(S^1) = x$. For any two pointed Hausdorff spaces X and Y, show that $\langle \Sigma X, Y \rangle$ is (naturally) bijective with $\langle X, \Omega Y \rangle$, where brackets means homotopy classes of based continuous maps.
Hint: This is a specific case of currying, or Eckmann-Hilton duality. This is extremely similar to the tensor-hom adjunction if you have seen this. Alternatively, this is saying that there are two ways of evaluating a continuous function on $X \wedge S^1$ to Y, and they are equivalent.

Vector Bundles and Parallelizability

Suggested Reading

There are a few different approaches and applications of vector bundles, and the information here is compiled from a few different sources. Hatcher's *Vector Bundles and K-Theory* chapter 1.1 covers most of the constructions and gives helpful examples, although the differential perspective is not emphasized. Lee's *Introduction to Smooth Manifolds* chapter 10 covers basics, but in far more technical detail than you need. I have also found that these notes below are quite useful http://staff.ustc.edu.cn/ wangzuoq/Courses/18F-Manifolds/Notes/Lec28.pdf

Vector bundles are an interesting intermediary between the differential theories and algebraic theories. On the one hand, the most common types of vector bundles are things like tangent bundles and normal bundles of manifolds, and canonical bundles of Grassmanians. On the other hand, the natural algebraic structure of a vector space makes the overall structure of a vector bundle somewhat tractable, and the theory of characteristic classes shows a deep relationship to cohomology. In this section, we denote a vector bundle as $V \to B$, for B our base and V the bundle. Sometimes, we use just V for short. Many times, we also use M instead of B to highlight that we work with manifolds. I wont go over the precise definitions again since you have probably seen these and since it isn't especially important.

You have probably seen different kinds of vector bundles within the 225 sequence. A few of the more common examples are given below.

Example. For any smooth manifold M, we can define the **tangent bundle** TM, whose fiber over $p \in M$ is the tangent space of p. This vector bundle has the same dimension as the manifold M.

Example. We can similarly define the **cotangent bundle** T^*M whose fiber over p is the dual vector space to T_pM.

Example. For a fixed embedding of the manifold M into Euclidean space $\mathbb{R}^n$ (or really to any other manifold), we can define the **normal bundle** of M to be the orthogonal complement of TM within $T\mathbb{R}^n|_M = M \times \mathbb{R}^n$. This is isomorphic to the quotient bundle $T\mathbb{R}^n|_M/TM$, although not naturally. This bundle represents the set of vectors pointing "outwards" from your manifold. This notion of outwards of course depends on the embedding itself, so this bundle is not intrinsic in the same way that the tangent bundle is. The dimension of this vector bundle is the same as the codimension of the embedding.

Example. The **k-degree exterior algebra** of the smooth manifold M is a vector bundle $\Omega^k(M)$. The fiber over p is the k-th exterior power $\Lambda^k(T_p^*M)$. The dimension as a vector bundle is $\binom{deg<}{k}$

Example. The open Mobius band is locally homeomorphic to $S^1 \times (0, 1)$. It in fact has the structure of a vector bundle (one dimensional) over S^1. This is not, however the trivial line bundle $S^1 \times \mathbb{R}$.

One of the big uses of vector bundles is that they allow us to formalize "smoothly choosing" a certain kind of vector across our manifold. We represent such a smooth choice as a smooth section $M \to V$. Recall that a section is a function $f : M \to V$ such that $proj \circ f = id_M$, i.e., that f sends each point p of M to some element within V_p.

Example. A smooth section of TM is a **vector field** on M.

Example. A smooth section of $\Omega^k(M)$ is a **degree k differential form on** M.

This language is especially useful if you are working with more general fiber bundles, although that is a bit far afield. This section is mostly focused on the following two definitions.

Definition. A smooth manifold embedded in $\mathbb{R}^n$ is **orientable** iff its normal bundle N has a nonvanishing smooth global section. This section represents a choice of outwards normal vector across the whole manifold. Equivalently, if the bundle of top dimensional differential forms $\Omega^n T^*M$ admits a non-vanishing section. Note that the first definition appears to be dependent on the embedding, but the second is not.

Definition. A smooth n-manifold M is **parallelizable** iff its tangent bundle T admits n smooth global sections that are linearly independent at each tangent space.

Exercise

Prove that the above definition is equivalent to the tangent bundle T being isomorphic (as a vector bundle) to the free bundle $M \times \mathbb{R}^n$.

Parallelizable manifolds are orientable. As an interesting historical fact, the only parallelizable spheres are S^0, S^1, S^3, and S^7. The proof for S^0 and S^1 is basically immediate. If you remember your Lie theory, the case for S^3 will follow from one of the qual problems below. The proof for S^7 requires the full general proof that uses normed division algebras like the octonions. Showing that these are the only ones is even harder.

Many of the exam problems require clever use of the following construction.

Definition. Given vector bundles V_1 and V_2 over the topological space B, define the **Whitney sum** $V_1 \oplus V_2$ is the vector bundle whose fiber at p is $V_{1,p} \oplus V_{2,p}$. The local trivializations are those inherited from V_1 and V_2, i.e. the ones that make this construction work like we want it to.

The funny property about addition of vector bundles is that vector bundles which do not look like the n dimensional trivial bundle $B \times \mathbb{R}^n$ can have a sum that *is* trivial in a larger dimension. Especially interesting is that there exist vector bundles V that are *not* trivial but whose sum with some trivial bundle $B \times \mathbb{R}^m$ is a trivial bundle $B \times \mathbb{R}^{m+k}$. These are called *stably trivial*.

Example. Let M be an orientable codimension 1 smooth manifold in $\mathbb{R}^n$. For example, M could be a sphere. Since M is orientable, the normal bundle M is the trivial bundle $M \times \mathbb{R}$. The tangent bundle T is usually not trivial, but we have $T \oplus N = T\mathbb{R}^n|_M = M \times \mathbb{R}^n$ by definition.

You can also essentially go in reverse;

Fact. If the base B is paracompact (like a manifold), V is a vector bundle on B, and V_0 is a vector subbundle, then there exists another subbundle $V_1 \subset V$ such that $V_0 \oplus V_1 \sim V$.

Formally, you would show this via Riemannian metrics (or the non-smooth equivalent for general spaces), although I believe you can take it for granted on the exam. Being able to push subbundles around sums is a generally useful thing to know how to do, so always keep in mind what the subbundles of your given bundle actually look like. In particular,

Fact. A nonvanishing global section of a vector bundle $V \to B$ is equivalent to a subbundle of V isomorphic to the trivial bundle $B \times \mathbb{R}$. More generally, a set of k linearly independent sections is equivalent to a subbundle isomorphic to $B \times \mathbb{R}^k$.

Note that not all one dimensional (line) bundles are trivial, so the above fact is slightly more powerful than you may be thinking.

These topics do not show up that often on the exams overall, but they have appeared much more often in recent years. I have really only scratched the surface of the amazing power of vector bundles, but I believe that this much should be enough to prepare you for the things that may appear on the exam.

S19 # 2

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be smooth, and let 0 be a regular value. Let $M = f^{-1}(0)$. Show that $M \times S^1$ is parallelizable.

F21 # 2

Show that the product of two spheres $S^p \times S^q$ is parallelizable provided p or q is odd.

F17 # 2, also F22 # 2

Show that Lie groups are parallelizable.

Bonus

Show that the aforementioned Mobius bundle over the circle is not isomorphic to the trivial one dimensional bundle.

Bonus

Show that the standard vector bundle projection $V \to^p B$ is a homotopy equivalence.

Genus g Surfaces

Suggested Reading

Hatcher describes the constructions in the most detail, starting on page 5. Guillemin and Pollack describes vector fields on these surfaces on page 125.

A very fundamental question at the heart of any mathematical field is *classifaction*, or trying to identify every single one of a certain family of mathematical objects. Classifying general manifolds up to diffeomorphism is unsolvable via incomputability of certain word problems. However, classification of closed (compact + boundariless) 2-dimensional manifolds *is* possible. Moreover, it is easy to describe and visualize all of them. This is the sort of result that makes geometry exam writers very giddy, because it means **you should be able to compute arbitrary data about closed surfaces**.

Theorem. A connected compact surface with boundary is diffeomorphic to one of the following:

1. The sphere S^2
2. A connected sum of tori T^2
3. A connected sum of projective planes $\mathbb{R}P^2$.

Any nonnegative number of boundary components (all necessarily circles) is allowed. For a general surface, the connected components are classified as above.

Fact. The Euler characteristic of an orientable genus g surface with n boundary components is $2-2g-n$. The Euler characteristic of the connect sum of p-many projective planes with n boundary components is $2-p-b$.

Therefore, the number of boundary components, Euler characteristic, and orientability invariants completely classify these manifolds. Here, the "connect sum" of surfaces A and B is the manifold obtained by cutting a small closed disk out of each of A and B, and then gluing together the cut manifolds along the boundary of the holes. We can understand visually that the location of the disk doesn't actually matter. It is known that the connect sum of $\mathbb{R}P^2$ and T^2 is the same as the connect sum of three real projective planes, hence why these families are exhaustive.

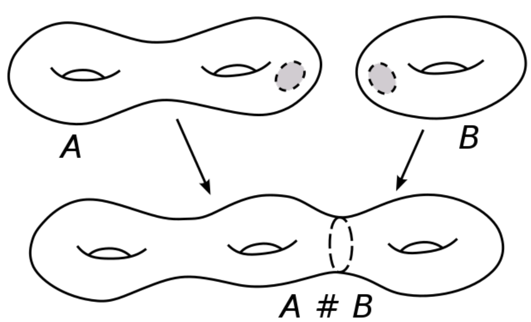
You have seen a few of these connect sums under different names.

Example. The connect sum of g torii is the orientable surface of genus g, i.e. the donut with g many holes. In some situations, it helps to visualize the holes as all being in the center. Sometimes, it is helpful to arrange them all in a row, similar to how they would occur in a connect sum. Sometimes still, you do a mix. We will see more of this last case later.

Example. The connect sum of two real projective planes is the Klein bottle.

I don't believe you will need to know higher connect sums of projective planes, but the other objects are very important. The rest of this section will focus on the genus g surfaces without boundary. In order to compute any algebraic invariants of these spaces, we will need some sort of decomposition. There are essentially two.

Example. By induction, the closed genus $g+1$ orientable surface is the connect sum of the torus and the closed genus g orientable surface. So, this space is covered by sets A' and T', with A' deformation retracting onto a punctured genus g surface, T' deformation retracting onto a punctured torus, and the intersection deformation retracting to a circle. Pictured here.

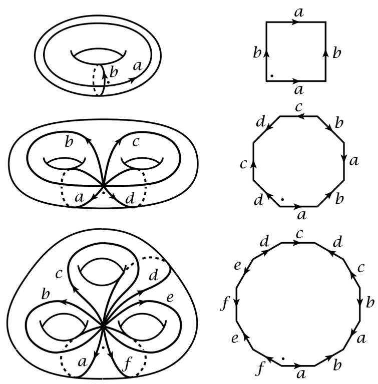


Connect sums of surfaces. Courtesy of Wikipedia

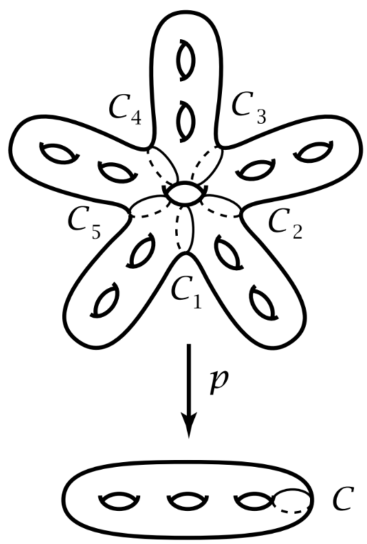
Example. One can also construct the closed genus g orientable surface as the quotient of the solid $2g$-gon as shown in the picture here. The case $g=1$ should be a familiar construction for the torus. This construction is described more carefully on Hatcher page 5.

There are a few other constructions worth knowing related to these surfaces.

Example. The closed genus $k(g-1)+1$ orientable surface is a k fold cover of the genus g surface. This is shown in the picture here. Understanding this visually is really all that is needed, although you may want to confirm that the details of covering space theory match up with the calculations that you do later on in this section. This construction is described more carefully on Hatcher page 73.



Folding regular polygons to make surfaces. Courtesy of Hatcher



Covering of one surface by another. Courtesy of Hatcher

Example. Imagine dunking the orientable closed genus g surface in water and quickly pulling it out. The water will begin to drip downwards along the surface. This yields a smooth vector field where the vector at position p is the direction of motion of water at p. This is pictured here. This seems like a very informal description, but this is exactly what Guillemin and Pollack describes on page 125, and is probably sufficient if you need an example of a special kind of vector field.

Many of the details about these objects can be described somewhat informally; you mostly need to know what the constructions *are* and then how to compute the important invariants from there.

Bonus (Related to S18 #7)

Compute the fundamental group of the closed orientable genus g surface.

A vector field on the closed orientable genus g surface. Courtesy of Guillemin and Pollack

Bonus

Compute the singular homology of the closed orientable genus g surface.

Bonus

Compute the Euler characteristic of the closed orientable genus g surface via the homology groups.

Bonus

Compute the Euler characteristic of the closed orientable genus g surface via Poincare-Hopf.

Bonus

Compute as much of the above as you can for the closed connect sum of p many real projective planes.

F13 # 7

Let $M = T^2 - D^2$ be the complement of a disk inside thet two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.
Sam's remark: I believe this disk needs to be open. Classification of the open surfaces is rather difficult.

S14 # 8

For $n \geq 2$, let X_n be the space obtained from a regular $2n$-gon by identifying opposite sides with parallel orientation.

1. Write down the associated cellular chain complex.
2. Show X_n is a surface and find its genus.

F14 # 7

A compact surface of genus g, smoothly embedded in $\mathbb{R}^3$, bounds a compact region called a handlebody H.

1. Prove that two copies of H glued along their boundaries produces a closed topological 3-manifold M.
2. Compute the homology of M.
3. Compute the relative homology of (M, H).

F17 # 7 (b)

If M is a surface of genus g, then what is min_v(# of zeros of v), where v ranges over vector fields whose zeros are isolated and have index ± 1?

F17 # 9

A compact surface (without boundary) of genus g, embedded into $\mathbb{R}^3$ in the standard way, bounds a compact 3 dimensional region called a handlebody H. Let $X = H \times \{0, 1, 2\}/\sim$, where $(x, i) \sim (x, j)$ for $x \in \partial H$, $i, j \in \{0, 1, 2\}$. Compute the homology of X.

S20 # 9

Consider $G_n = \langle a_1, b_1, a_2, b_2, ...a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1} \rangle$. For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

F20 #9

Let Σ_5 be a compact oriented surface of genus 5 without boundary. Does there exist an immersion $f : T^2 \to \Sigma_5$? Justify your answer.

S23 # 6b

Let $X = \Sigma_g$ be a closed genus g surface. What path-connected, closed surfaces can cover X?

Group Theory

Suggested Reading

Group theory is a rabbit hole to which there is no bottom. Learning group theory usually requires 2/3 of a quarter's algebra sequence, and learning it *well* requires at least another quarter after that. It isn't reasonable for you to try and learn "all" group theory just to prepare for the strange questions which sometimes appear on the exam. You should view this section as more of a warning than a true guide. In any case, Lang's *Algebra* and Dummit & Foote's *Abstract Algebra* have some decent exercises between them, though each has its own detractors. Hatcher section 1.3 has a good review of the covering space theory.

In very recent years, there have been more and more group theory questions disguised as covering space questions. The key result needed in these problems is the following, which is certainly taught in the 225 sequence. This is Hatcher theorems 1.38, 1.39.

Theorem. Let X be path-connected, locally path-connected, and semilocally simply-connected. Then

1. There is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\overline{X}, \overline{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. Denote $H := p_*(\pi_1(\overline{X}, \overline{x_0})) \subset \pi_1(X, x_0)$.
2. There is a bijection between the number of sheets of the cover $(\overline{X}, \overline{x_0})$ and the index of H.
3. $(\overline{X}, \overline{x_0})$ is normal iff H is normal
4. The group of deck transformations $G(\overline{X})$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$. In particular, $G(\overline{X})$ is isomorphic to $\pi_1(X, x_0)/H$ when $\overline{X}$ is normal.

Covering spaces are incredibly important within topology, and such a strong classificatory result allows us to freely transpose topology problems into algebra and vice versa. For example, one can prove that subgroups of finitely generated free groups are also free via looking at graphs with these groups as π_1 (see Hatcher 1.A). The downside of this (for you) is that arbitrarily difficult problems in group theory can be transposed into topology for no reason other than to check if you know this classification.

Bonus (Don't actually try this)

Let X be a pclpcssc space for which $\pi_1(X)$ is a finite group of odd order. Show that there exists a sequence of covering spaces $\overline{X_0} \to \overline{X_1} \to \overline{X_2} ... \overline{X_n} = X$ such that

1. $\overline{X_0}$ is the universal cover of X
2. Each cover $\overline{X_i} \to^{p_i} \overline{X_{i+1}}$ is an abelian cover (normal and $\pi_1(\overline{X_{i+1}})/p_i^*(\pi_1(\overline{X_i}))$ is abelian)

The topological properties that are desirable for covering spaces are "simply-connected", "normal", "finite(-sheeted)", and rarely "abelian". So, many of the key uses of this correspondence are to study the fundamental group and determine what conditions can be imposed to guarantee that path-connected covers have these properties. There is no general way to prepare for all possible group theory tricks, but I have included a few useful results along these lines. Some of these are fairly easy, but are easy to forget. Some are just decent practice with these ideas.

Bonus (D&F 3.1.22.b)

Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup.

Bonus

A subgroup H of G is called *verbal* if it is generated by all elements that can be formed by substituting group elements for variables in a given set of words. For example, the commmutator subgroup is generated by all elements of the form $ghg^{-1}h^{-1}$. Prove that verbal subgroups are normal.

Bonus

Let G be a finite group of order n, and p the smallest prime dividing n. Prove that any subgroup of index p is normal. (Note that this is always the case if $p = 2$!)

Bonus (D&F 4.4.7)

Let H be the unique subgroup of a given order in a group G. Show that H is normal in G.

Bonus (D&F 6.1.21)

For any group G, the Frattini subgroup of G (denoted by $\Phi(G)$) is defined to be the intersection of all maximal (proper) subgroups of G (if G has no maximal subgroups, set $\Phi(G) = G$). Prove that $\Phi(G)$ is normal in G.

Bonus (D&F 6.1.26(a)) (a bit of Sylow theory required)

Let $\Phi(G)$ be defined as above. Suppose G has order p^n for some prime p. Prove that $P/\Phi(G)$ is abelian.

Bonus (Lang 1.9, also D&F 4.2.8)

a Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index. (Hint: If $(G : H) = n$, find a homomorphism of G into S_n whose kernel is contained in H.)
b Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Actual group theory is not really emphasized within the 225 curriculum (justifiably), hence its appearance in this guide. Of course, many qual problems require you to compute some sort of fundamental group or homology group. The point of this section is that you often need group *theory* beyond topological results/correspondences in order to answer the topological questions, and infrequently the reverse. The questions below have roughly this theme. Hypothetically, some of these can be done *without* using group theoretic results, and instead directly proving something about covering spaces. If you can find such a proof, you should probably try and remember it well, since those techniques are more transferable to the other problems than is strict group theory.

F18 # 6

Can a finite rank free group have a finite index subgroup of smaller rank?

S19(a) # 8, very similar to F19 #10b, S18 #10a, Hatcher 1.3.9

Show that any continuous map $\mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotopic

S21 # 2

a Let $F : S^n \to S^n$ be a continuous map. Show that if F has no fixed point, then the degree of the map, deg $F = (-1)^{n+1}$.
b Show that if X has S^{2n} as a universal covering space, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

F21 # 8 (see # 7 for a hint)

Let M be a connected non-orientable manifold whose fundamental group G is simple (that is, has no non-trivial normal subgroup). Prove that G must be isomorphic to $\mathbb{Z}/2$.

F22 # 8, also Hatcher 1.3.18. See also Lang 1.3

Let X be a pclpcssc space. Recall that a path connected covering space $\overline{X} \to X$ is abelian if $\pi_1(X)$ is normal in $\pi_1(X)$ and the quotient is abelian. Show that there is a universal abelian cover: this is an abelian cover $\overline{X} \to X$ such that for any other abelian cover $\overline{Y} \to X$, there is a covering map $\overline{X} \to \overline{Y}$ factoring the map $\overline{X} \to X$.

S23 # 7

A group G is divisible if for all n, the map $g \to g^n$ from G to itself is surjective. Show that if X is a path-connected CW-complex and if $\pi_1(X, x)$ is a divisible group, then the only path-connected finite cover of X is X itself. (Hint: This can be proven directly or by first showing that a divisible group has no finite index subgroups.)

Bonus (Hatcher 1.A.7). See also Hatcher 1.A.8-1.A.13

If F is a finitely generated gree group and N is a nontrivial normal subgroup of infinite index, show, using covering spaces, that N is not finitely generated.

Assorted Trickery

This section features any other strange tricks or techniques that are either needed for some proofs or at least make things much easier. Some of these may warrant their own sections in the future.

F17 # 4, also S20 # 3

Let $\omega = xdy - ydx - dz$. Prove that $f\omega$ is not closed for any nowhere zero smooth f on $\mathbb{R}^3$.

Proof. First, we show that $\omega \wedge d\omega$ is nowhere zero. We compute that $d\omega = 2dxdy$. Then $\omega \wedge d\omega = -2dxdydz$, which is nowhere zero.

Now, call a differential 1 form on $\mathbb{R}^3$ a contact form if $\eta \wedge d\eta$ is nowhere zero. ω as above is a contact form. We prove that for η a contact form, so is $f\eta$ for f smooth and nowhere zero. We see that $f\eta \wedge d(f\eta) =$

$f\eta \wedge (df \wedge \eta + f \wedge d\eta))$. $\eta \wedge \eta = 0$ since η is a one form (this does not work for even-dimensional forms!). The remaining term is $f^2\eta \wedge d\eta$, which is nowhere zero by assumption.

So, $f\omega$ is not closed for such f, since $(f\omega) \wedge d(f\omega) \neq 0$ anywhere. □

This method of proof may appear mysterious to you, but is worth remembering. I suggest googling "contact geometry", and observing that ω is the canonical example of a contact form (up to some sign change). Ko Honda is a contact geometer, so this is likely the sort of proof he had in mind. An explicit computation in coordinates is also possible for this problem, but not recommended.

F15 # 8, also S20 # 5

Show that $\mathbb{C}P^{2n}$ is not a covering for any manifold other than itself.

Proof. Note that $\pi_1(\mathbb{C}P^{2n}) = 0$, since this space has a CW decomposition without 1-cells. If $\mathbb{C}P^{2n}$ covers a manifold X, we will show that $\pi_1(X) = 0$ as well, since by the Galois correspondence, this forces $X \sim \mathbb{C}P^{2n}$. Note that $\pi_1(X)$ acts freely on $\mathbb{C}P^{2n}$, meaning every non-identity element has no fixed points. So, we show that every such induced map on $\mathbb{C}P^{2n}$ must have a fixed point. We prove more generally that any continuous map f from $\mathbb{C}P^{2n}$ to itself has a fixed point.

The Lefschetz Trace Formula (valid since $\mathbb{C}P^{2n}$ is compact) gives that $L(f) = \sum_{k\geq 0}(-1)^k tr(f^* : H^k(\mathbb{C}P^{2n}, \mathbb{Q}) \to H^k(\mathbb{C}P^{2n}, \mathbb{Q}))$. For $\mathbb{C}P^{2n}$, these spaces vanish in odd dimension, and are $\mathbb{Q}$ in even dimension no greater than $4n$. Moreover, via the cohomology ring for $\mathbb{C}P^{2n}$, if x is a generator for $H^2(\mathbb{C}P^{2n}, \mathbb{Q})$, then x^i is a generator for $H^{2i}(\mathbb{C}P^{2n}, \mathbb{Q})$. So, since f^* is a ring homomorphism on the cohomology ring, if $f^*(x) = rx$, $f^*(x^i) = r^i x^i$. Therefore, $L(f) = 1 + r + r^2 + ...r^{2n}$. If $r = 1$, this is $2n + 1 \neq 0$. Otherwise, this is $\frac{1-r^{2n+1}}{1-r}$. Since r is rational, $r^{2n+1} = 1$ implies $r = 1$, so $L(f)$ is nonzero. By the Lefschetz fixed point theorem, f has a fixed point. Therefore, $\pi_1(X)$ must be trivial, and so $X \sim \mathbb{C}P^{2n}$.

□

This problem looks scary, and can be tricky since this is really not at all a question about covering spaces. You should consider LFPT whenever you see an even dimensional complex projective space. To my knowledge, this is the only sort of space for which the trace formula is even a feasible computation, let alone a useful one.